Calculus 2

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Calculus 2

Lecture 9

Power Series

Power Series

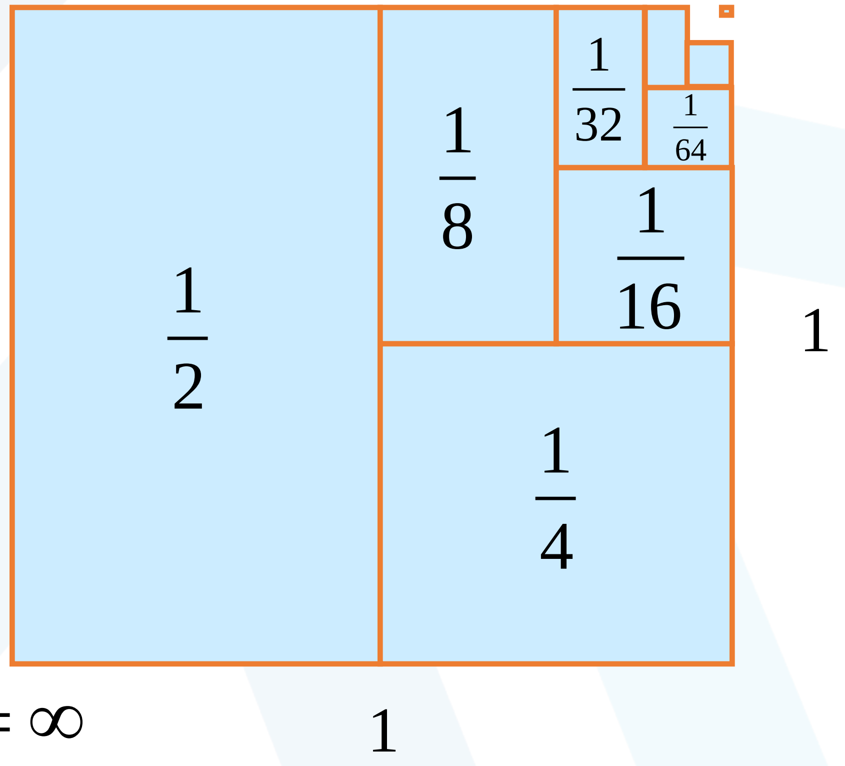
INTRODUCTION

Now that we can test many infinite series of numbers for convergence, we can study sums that look like "infinite polynomials." We call these sums *power series* because they are defined as infinite series of powers of some variable, in our case x. Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series. With power series we can extend the methods of calculus to a vast array of functions, making the techniques of calculus applicable in an even wider setting

Power Series

Start with a square one unit by one unit:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$



This is an example of an <u>infinite series</u>.

This series <u>converges</u> (approaches a limiting value.)

Many series do not converge: $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots=\infty$

Geometric Series

In a geometric series, each term is found by multiplying the preceding term by the same number, r.

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

This converges to $\frac{a}{1-r}$ if $|r| < 1$, and diverges if $|r| \geq 1$.

$-1 < r < 1$ is the interval of convergence.

Geometric Series

The partial sum of a geometric series is: $$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

If $|r|<1$ then $$\lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} = \frac{a}{1-r}$$

If $|x|<1$ and we let $r = x$, then:

$$1+x+x^2+x^3+\cdots$$

The more terms we use, the better our approximation (over the interval of convergence.)

Power Series

A power series is in this form: $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$$

or $$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots + c_n (x-a)^n + \cdots$$

The coefficients $c_0, c_1, c_2 \ldots$ are constants.

The center "a" is also a constant.

(The first series would be centered at the origin if you graphed it. The second series would be shifted left or right. "a" is the new center.)

Power Series

Once we have a series that we know, we can find a new series by doing the same thing to the left and right hand sides of the equation.

Example 1: $\frac{1}{1+x}$ This is a geometric series where $r=-x$.

$$\frac{1}{1+x}=1-x+x^2-x^3+\cdots$$

To find a series for $\frac{x}{1+x}$ multiply both sides by x. $\frac{x}{1+x}=x-x^2+x^3-x^4\cdots$

Power Series

Example 2: Given: $$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$ find: $$\frac{1}{(1-x)^2}$$

$$\frac{d}{dx}\frac{1}{1-x}=\frac{d}{dx}(1-x)^{-1}=-(1-x)^{-2}\cdot -1=\frac{1}{(1-x)^2}$$

So: $$\frac{1}{(1-x)^2}=\frac{d}{dx}\left(1+x+x^2+x^3+\cdots\right)=1+2x+3x^2+4x^3+\cdots$$

We differentiated term by term.

Power Series

Example 3: Given: $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ find: $\ln(1+x)$

$$\int \frac{1}{1+x} dx = \ln(1+x) + c$$

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

Power Series

Example 4:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(1 - t + t^2 - t^3 + \cdots\right) dt$$

$$\ln(1+t)\Big|_0^x = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + \cdots\Big|_0^x$$

$$\ln(1+x) - \ln(1+0) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \qquad -1 < x < 1$$

Power Series

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdot \cdot \cdot + a_n(x-c)^n + \cdot \cdot \cdot$$

THEOREM Convergence of a Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists a real number $R > 0$ such that the series converges absolutely for

$$|x-c| < R$$

and diverges for

$$|x-c| > R.$$

Representation of Functions by Power Series

Geometric Power Series

In this section and the next, you will study several techniques for finding a power series that represents a function. Consider the function

$$f(x) = \frac{1}{1-x}.$$

The form of f closely resembles the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

In other words, when $a = 1$ and $r = x$, a power series representation for $1/(1-x)$, centered at 0, is

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} x^n \\ &= 1 + x + x^2 + x^3 + \cdot \cdot \cdot, \quad |x| < 1. \end{aligned}$$

EXAMPLE

Find a power series for $f(x) = \frac{1}{x}$, centered at 1.

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Solution Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{1}{x} = \frac{1}{1 - (-x + 1)} = \frac{a}{1 - r}$$

which implies that $a = 1$ and $r = 1 - x = -(x - 1)$. So, the power series for $f(x)$ is

$$\begin{aligned} \frac{1}{x} &= \sum_{n=0}^{\infty} ar^n \\ &= \sum_{n=0}^{\infty} [-(x - 1)]^n \\ &= \sum_{n=0}^{\infty} (-1)^n (x - 1)^n \\ &= 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdot \cdot \cdot . \end{aligned}$$

This power series converges when

$$|x - 1| < 1$$

EXAMPLE

Find a power series for

$$f(x) = \ln x$$

centered at 1.

Solution From Example 2, you know that

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n. \qquad \text{Interval of convergence: } (0, 2)$$

Integrating this series produces

$$\ln x = \int \frac{1}{x}\,dx + C$$
$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.$$

By letting $x = 1$, you can conclude that $C = 0$. Therefore,

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$
$$= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots. \qquad \text{Interval of convergence: } (0, 2]$$

EXAMPLE

Find a power series for

$$f(x) = \frac{3x - 1}{x^2 - 1}$$

Solution Using partial fractions, you can write $f(x)$ as

$$\frac{3x - 1}{x^2 - 1} = \frac{2}{x + 1} + \frac{1}{x - 1}.$$

By adding the two geometric power series

$$\frac{2}{x + 1} = \frac{2}{1 - (-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x - 1} = \frac{-1}{1 - x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

you obtain the power series shown below.

$$\frac{3x - 1}{x^2 - 1} = \sum_{n=0}^{\infty} [2(-1)^n - 1]x^n$$
$$= 1 - 3x + x^2 - 3x^3 + x^4 - \cdot \cdot \cdot$$

Taylor and Maclaurin Series

THEOREM **The Form of a Convergent Power Series**

If f is represented by a power series $f(x) = \Sigma\, a_n(x - c)^n$ for all x in an open interval I containing c, then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots.$$

Taylor and Maclaurin Series

Definition of Taylor and Maclaurin Series

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at c.** Moreover, if $c = 0$, then the series is the **Maclaurin series for f.**

Form the Maclaurin series for $f(x) = \sin x$

SOLUTION

$$f(x) = \sin x \qquad f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \qquad f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x \qquad f''(0) = -\sin 0 = 0$$

$$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -\cos 0 = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = \sin 0 = 0$$

$$f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = \cos 0 = 1$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \ldots$$

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\dots \quad = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

EXAMPLE

Find the Taylor series for $y = \cos(x)$ at $x = \frac{\pi}{2}$

$$f(x) = \cos x \quad f\left(\frac{\pi}{2}\right) = 0$$

$$f'(x) = -\sin x \quad f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(x) = -\cos x \quad f''\left(\frac{\pi}{2}\right) = 0$$

$$f'''(x) = \sin x \quad f'''\left(\frac{\pi}{2}\right) = 1$$

$$f^{(4)}(x) = \cos x \quad f^{(4)}\left(\frac{\pi}{2}\right) = 0$$

$$P(x) = 0 - 1\left(x - \frac{\pi}{2}\right) + \frac{0}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{2}\right)^3 + \cdots$$

$$P(x) = -\left(x - \frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^3}{3!} - \frac{\left(x - \frac{\pi}{2}\right)^5}{5!} + \cdots$$

EXAMPLE

Find the Maclaurin series for

$$f(x) = \cos(2x)$$

SOLUTION Rather than start from scratch, we can use the function that we already know:

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \quad \cdots$$

$$\cos(2x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \frac{(2x)^{10}}{10!} \quad \cdots$$

EXAMPLE

Find the Maclaurin series for $f(x) = \frac{1}{1-x}$

$f^{(n)}(x)$	$f^{(n)}(0)$
$(1-x)^{-1}$	1
$(1-x)^{-2}$	1
$2(1-x)^{-3}$	2
$6(1-x)^{-4}$	$6 = 3!$
$24(1-x)^{-5}$	$24 = 4!$

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\frac{1}{1-x} = 1 + 1x + \frac{2}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4 + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with $a = 1$ and $r = x$.

EXAMPLE Find the Maclaurin series for

$$f(x) = \ln(1+x)$$

$f^{(n)}(x)$	$f^{(n)}(0)$
$\ln(1+x)$	0
$(1+x)^{-1}$	1
$-(1+x)^{-2}$	-1
$2(1+x)^{-3}$	2
$-6(1-x)^{-4}$	$-6 = -3!$

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\ln(1+x) = 0 + 1x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

EXAMPLE

Find the Maclaurin series for $f(x) = e^x$

$f^{(n)}(x)$	$f^{(n)}(0)$
e^x	1
e^x	1
e^x	1
e^x	1
e^x	1

$$P(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$e^x = 1 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

EXAMPLE

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Substitute xi for x.

$$e^{xi} = 1 + xi + \frac{(xi)^2}{2!} + \frac{(xi)^3}{3!} + \frac{(xi)^4}{4!} + \frac{(xi)^5}{5!} + \frac{(xi)^6}{6!} + \cdots$$

$$e^{xi} = 1 + xi + \frac{x^2 i^2}{2!} + \frac{x^3 i^3}{3!} + \frac{x^4 i^4}{4!} + \frac{x^5 i^5}{5!} + \frac{x^6 i^6}{6!} + \cdots$$

$$e^{xi} = 1 + xi - \frac{x^2}{2!} - \frac{x^3 i}{3!} + \frac{x^4}{4!} + \frac{x^5 i}{5!} - \frac{x^6}{6!} + \cdots$$

Factor out the i terms.

$$e^{xi} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$e^{xi} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

This is the series for cosine. This is the series for sine.

EXAMPLE

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Use a power series to approximate

$$\int_0^1 e^{-x^2}\,dx$$

with an error of less than 0.01.

Solution Replacing x with $-x^2$ in the series for e^x produces the following.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$\int_0^1 e^{-x^2}\,dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

Summing the first four terms, you have

$$\int_0^1 e^{-x^2}\,dx \approx 0.74$$

which, by the Alternating Series Test, has an error of less than $\frac{1}{216} \approx 0.005$.

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \leq 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots \qquad -\infty < x < \infty$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots \qquad -1 \le x \le 1$$

$$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots \qquad -1 \le x \le 1$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots \qquad -1 < x < 1^*$$

Thank you for your attention