Lecture 1: Systems of Linear Equations

CECC122: Linear Algebra and Matrix Theory

Manara University

2023-2024

جـامعة
المَـنارة
MANARA UNIVERSITY

1.1 Introduction to Systems of Linear Equations

- a linear equation in *n* variables: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

  $a_1, a_2, \ldots, a_n, b$: real numbers

  a_1: leading coefficient

  x_1: leading variable

- Notes:

(1) Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

- **Ex 1: (Linear or Nonlinear)**

Linear $(a)\ 3x + 2y = 7$

$(b)\ \frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

Linear $(c)\ x_1 - 2x_2 + 10x_3 + x_4 = 0$

$(d)\ (\sin\frac{\pi}{2})x_1 - 4x_2 = e^2$ Linear

NonLinear $(e)\ xy + z = 2$ — Products

$(f)\ e^x - 2y = 4$ NonLinear — Exponential

NonLinear $(g)\ \sin x_1 + 2x_2 - 3x_3 = 0$ — Trigonometric functions

$(h)\ \frac{1}{x} + \frac{1}{y} = 4$ NonLinear — Not the first power

- a solution of a linear equation in n variables:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ such that: $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$

- Solution set: the set of all solutions of a linear equation
- Ex 2: (Parametric representation of a solution set)

$x_1 + 2x_2 = 4$ (2, 1) is a solution, i.e. $x_1 = 2, x_2 = 1$

If you solve for x_1 in terms of x_2, you obtain $x_1 = 4 - 2x_2$

By letting $x_2 = t$ you can represent the solution set as $x_1 = 4 - 2t$

And the solutions are $\{(4 - 2t,\ t) | t \in R\}$ or $\{(s,\ 2 - ½s) | s \in R\}$

In vector form: $(x_1, x_2) = (4, 0) + t(-2, 1) = (0, 2) + s(1, -½)$

- a system of m linear equations in n variables:

$$\begin{aligned}
a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 &+ a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
a_{m1}x_1 &+ a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}$$

- Consistent:

A system of linear equations has <u>at least one solution</u>.

- Inconsistent:

A system of linear equations has <u>no solution</u>.

- Ex 3: (Solution of a system of linear equations)

$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$

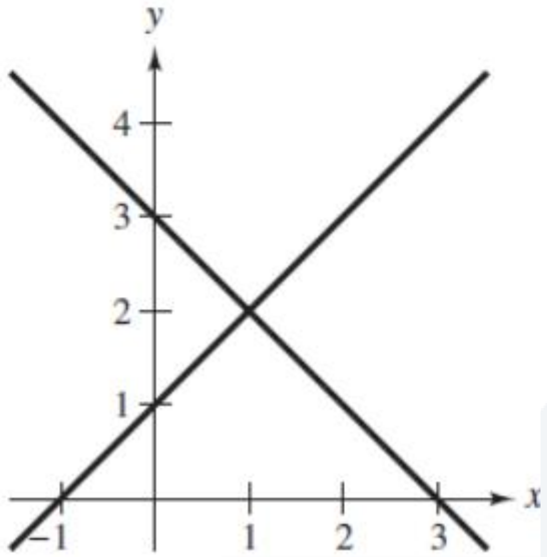
two intersecting lines

$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

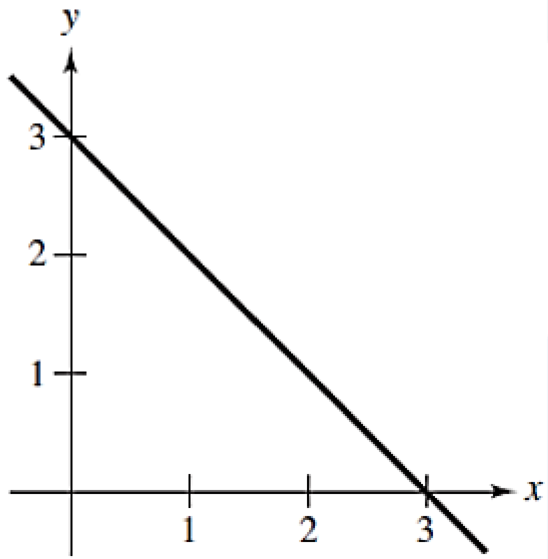
two coincident lines

$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

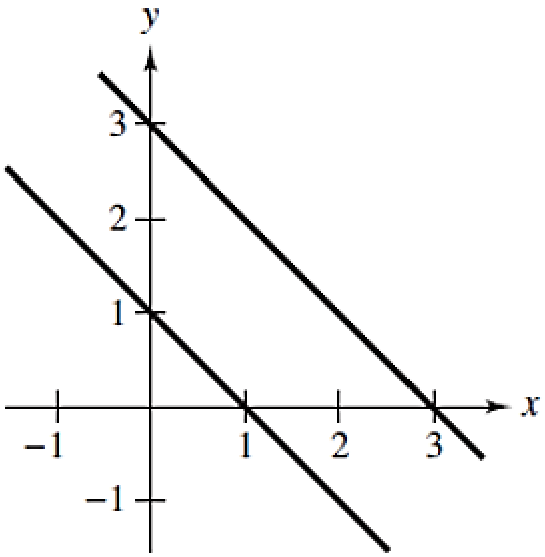
two parallel lines



exactly one solution



infinite number



no solution

- Ex 4: (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y &= 5 \quad (1) \\ y &= -2 \quad (2) \end{aligned}$$

Sol: By substituting $y = -2$ into (1), you obtain

$$\begin{aligned} x - 2(-2) &= 5 \\ x &= 1 \end{aligned}$$

The system has exactly one solution: $x = 1, y = -2$

- Ex 5: (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1)\\ y + 3z &= 5 \quad (2)\\ z &= 2 \quad (3)\end{aligned}$$

Sol: Substitute $z = 2$ into (2)

$$\begin{aligned} y + 3(2) &= 5\\ y &= -1\end{aligned}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{aligned} x - 2(-1) + 3(2) &= 9\\ x &= 1\end{aligned}$$

The system has exactly one solution: $x = 1,\ y = -1,\ z = 2$

- **Equivalent:**

Two systems of linear equations are called equivalent if they have precisely the same solution set

- **Notes:**

Each of the following operations on a system of linear equations produces an equivalent system.

(1) Interchange two equations.

(2) Multiply an equation by a nonzero constant.

(3) Add a multiple of an equation to another equation.

- Ex 6: Solve a system of linear equations (consistent system)

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ -x + 3y &= -4 \quad (2) \\ 2x - 5y + 5z &= 17 \quad (3) \end{aligned}$$

Sol: $(1)+(2) \rightarrow (2)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \quad (4) \\ 2x - 5y + 5z &= 17 \end{aligned}$$

$(1)\times(-2)+(3) \rightarrow (3)$

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \quad (5) \end{aligned}$$

$$(4) + (5) \rightarrow (5)$$

$$\begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \qquad (6) \end{array}$$

$$(6) \times \frac{1}{2} \rightarrow (6)$$

$$\begin{array}{rcrcrcl} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$$

So the solution is: $x = 1,\ y = -1,\ z = 2$

- Ex 7: Solve a system of linear equations (inconsistent system)

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \quad (1) \\ 2x_1 - x_2 - 2x_3 &= 2 \quad (2) \\ x_1 + 2x_2 - 3x_3 &= -1 \quad (3) \end{aligned}$$

Sol: $(1)\times(-2)+(2)\rightarrow(2)$

$(1)\times(-1)+(3)\rightarrow(3)$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \quad (4) \\ 5x_2 - 4x_3 &= -2 \quad (5) \end{aligned}$$

$$(4) \times (-1) + (5) \rightarrow (5)$$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ \boxed{0 = -2} \end{aligned}$$

(a false statement)

So the system has no solution (an inconsistent system).

Ex 8: Solve a system of linear equations (infinitely many solutions)

$$\begin{array}{rcrcrcrl} & & x_2 & - & x_3 & = & 0 & (1)\\ x_1 & & & - & 3x_3 & = & -1 & (2)\\ -x_1 & + & 3x_2 & & & = & 1 & (3)\end{array}$$

Sol: $(1) \leftrightarrow (2)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & (1)\\ & & x_2 & - & x_3 & = & 0 & (2)\\ -x_1 & + & 3x_2 & & & = & 1 & (3)\end{array}$$

$(1)+(3) \rightarrow (3)$

$$\begin{array}{rcrcrcrl} x_1 & & & - & 3x_3 & = & -1 & \\ & & x_2 & - & x_3 & = & 0 & \\ & & 3x_2 & - & 3x_3 & = & 0 & (4)\end{array}$$

جامعة المنارة
MANARA UNIVERSITY

$(2)\times(-3)+(4)\rightarrow(4)$

$$\begin{aligned} x_1 \qquad\quad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ \boxed{0 = 0} \end{aligned}$$

(a True statement)

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

letting $x_3 = t$, then the solutions are

$$\{(3t-1,\ t,\ t) | t \in R\}$$

So the system has infinitely many solutions.

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- $m \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \; m \text{ rows}$$

n columns

- Notes:

(1) Every entry a_{ij} in a matrix is a number.

(2) A matrix with m rows and n columns is said to be of size $m \times n$.

(3) If $m = n$, then the matrix is called square of order n.

(4) For a square matrix, $a_{11}, a_{22}, \ldots, a_{nn}$ are called the main diagonal entries.

- **Ex 1:**

Matrix	Size
$\begin{bmatrix} 2 \end{bmatrix}$	1×1
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	2×2
$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$	1×4
$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$	3×2

- a system of m equations in n variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Matrix form: $A\boldsymbol{x} = \boldsymbol{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- **Augmented matrix:**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & \big| & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \big| & b_2 \\ & \vdots & & & \big| & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & \big| & b_m \end{bmatrix} = [A \mid \boldsymbol{b}]$$

- **Coefficient matrix:**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = A$$

- **Elementary row operation:**

(1) Interchange two rows. $r_{ij} : R_i \leftrightarrow R_j$

(2) Multiply a row by a nonzero constant. $r_i^{(k)} : (k)R_i \rightarrow R_i$

(3) Add a multiple of a row to another row. $r_{ij}^{(k)} : (k)R_i + R_j \rightarrow R_j$

- **Row equivalent:**

Two matrices are said to be row equivalent if one can be obtained from the other by a finite sequence of **elementary row operation**.

- Row-echelon form: (1, 2, 3)
- Reduced row-echelon form: (1, 2, 3, 4)

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).

(3) For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1.

- Ex 4: (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ row-echelon form

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row-echelon form

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row-echelon form

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

- **Gaussian elimination:**

  The procedure for reducing a matrix to a row-echelon form.

- **Gauss-Jordan elimination:**

  The procedure for reducing a matrix to a reduced row-echelon form.

- **Notes:**

  (1) Every matrix has an unique reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.

  (Different sequences of row operations can produce different row-echelon forms.)

Ex 5: Solve a system by Gauss-Jordan elimination method (one solution)

$$\begin{aligned} x &- 2y + 3z = 9 \\ -x &+ 3y = -4 \\ 2x &- 5y + 5z = 17 \end{aligned}$$

Sol: augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\ r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_3^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{31}^{(-3)},\ r_{32}^{(-3)},\ r_{21}^{(2)}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

row-echelon form

reduced row-echelon form

- Ex 7: Solve a system by G.J. elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 &= 1 \end{aligned}$$

Sol:

augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\ r_{12}^{(-3)},\ r_2^{(-1)},\ r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

reduced row-echelon form

$$\longrightarrow \begin{aligned} x_1 + 5x_3 &= 2 \\ x_2 - 3x_3 &= -1 \end{aligned}$$

leading variables: x_1, x_2

free variable: x_3

$$
\begin{aligned}
x_1 &= 2 - 5x_3 \\
x_2 &= -1 + 3x_3
\end{aligned}
$$

letting $x_3 = t$, then the solutions are:

$$\{(2 - 5t, -1 + 3t, t) | t \in R\}$$

So the system has infinitely many solutions.

- Ex 8: Solve a system by Gauss-Jordan elimination method (no solution)

$$
\begin{aligned}
x_1 - x_2 + 2x_3 &= 4 \\
x_1 + x_3 &= 6 \\
2x_1 - 3x_2 + 5x_3 &= 4 \\
3x_1 + 2x_2 - x_3 &= 1
\end{aligned}
$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \xrightarrow{r_{12}^{(-1)},\, r_{13}^{(-2)},\, r_{14}^{(-3)},\, r_{23}^{(1)}} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix}$$

$$\longrightarrow \begin{array}{rcl} x_1 - x_2 + 2x_3 & = & 4 \\ x_2 - x_3 & = & 2 \\ 0 & = & -2 \\ 5x_2 - 7x_3 & = & -11 \end{array}$$

Because the third equation is not possible, the system has no solution.