Exercises 1: Systems of Linear Equations

CECC122: Linear Algebra and Matrix Theory

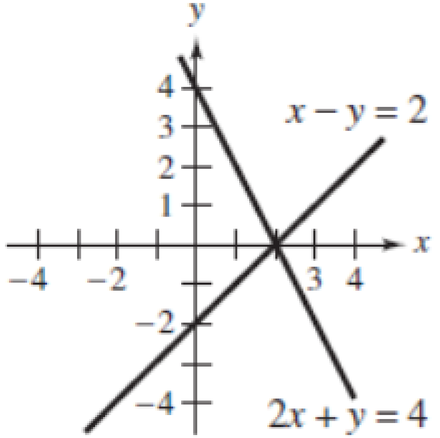
Manara University

2023-2024

Graph the system of linear equations. Solve the system and interpret your answer

①

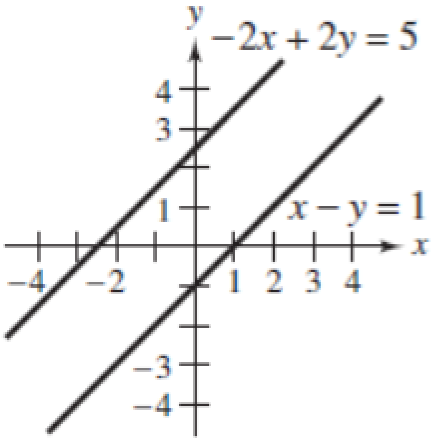
$$\begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$$



Adding the first equation to the second produces a new equation, $3x = 6$, or $x = 2$. So, $y = 0$, and the solution is $x = 2$, $y = 0$.

②

$$\begin{aligned} x - y &= 1 \\ -2x + 2y &= 5 \end{aligned}$$

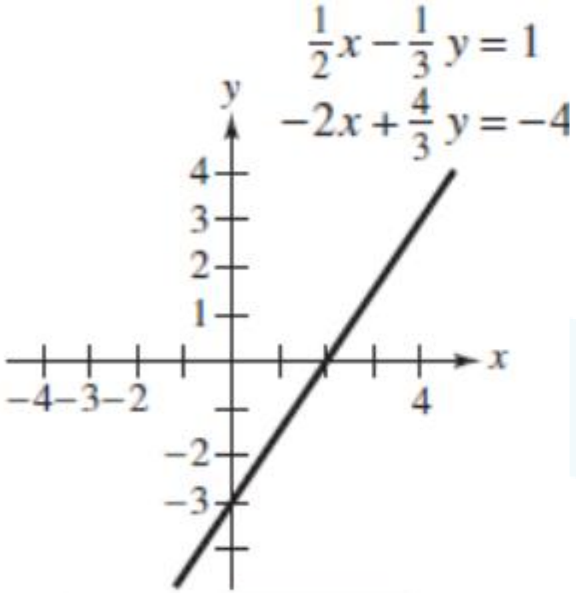


Adding 2 times the first equation to the second produces

$$\begin{aligned} x - y &= 1 \\ 0 &= 7 \end{aligned}$$

The second equation is a false statement, therefore the original system has no solution. The two lines are parallel

③
$$\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 1 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$$



Multiplying the first equation by 2 produces

$$\begin{aligned} x - \tfrac{2}{3}y &= 1 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$$

Adding 2 times the first equation to the second equation produces

$$\begin{aligned} x - \tfrac{2}{3}y &= 2 \\ 0 &= 0 \end{aligned}$$

Choosing $y = t$ as the free variable, $x = (2/3)t + 2$.
So, the solution set is $x = (2/3)t + 2$ and $y = t$, where t is any real number.

Solve the system of linear equations

①
$$\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= -1 \end{aligned}$$

Adding –3 times the first equation to the second equation produces

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_2 &= -1 \end{aligned}$$

Using back-substitution you can conclude that the system has exactly one solution: $x_1 = -1$ and $x_2 = -1$

②
$$\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 14 \end{aligned}$$

Adding –2 times the first equation to the second equation produces

$$\begin{aligned} 3x + 2y &= 2 \\ 0 &= 10 \end{aligned}$$

Because the second equation is a false statement, the original system of equations has no solution.

③
$$\begin{aligned} \tfrac{2}{3}x + \tfrac{1}{6}y &= 0 \\ 4x + y &= 0 \end{aligned}$$

Multiplying the first equation by 3/2 produces
$$\begin{aligned} x + \tfrac{1}{4}y &= 0 \\ 4x + y &= 0 \end{aligned}$$

Adding –4 times the first equation to the second produces
$$\begin{aligned} x + \tfrac{1}{4}y &= 0 \\ 0 &= 0 \end{aligned}$$

Choosing $x = t$ as the free variable, $y = -(1/4)t$

So the solution set is $x = t$ and $y = -(1/4)t$, where t is any real number

$$④\quad \begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 3x \quad\quad - z &= 0 \end{aligned}$$

Adding –2 times the first equation to the second produces

$$\begin{aligned} x + y + z &= 6 \\ -3y - z &= -9 \\ -3y - 4z &= -18 \end{aligned}$$

Dividing the second equation by –3 produces

$$\begin{aligned} x + y + z &= 6 \\ y + \tfrac{1}{3}z &= 3 \\ -3y - 4z &= -18 \end{aligned}$$

Adding 3 times the second equation to the third equation produces

$$\begin{aligned} x + y + z &= 6 \\ y + \tfrac{1}{3}z &= 3 \\ -3z &= -9 \end{aligned}$$

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Dividing the third equation by -3 produces

$$\begin{aligned} x + y + z &= 6 \\ y + \tfrac{1}{3}z &= 3 \\ z &= 3 \end{aligned}$$

Using back-substitution you can conclude that the system has exactly one solution: $x = 1$, $y = 2$, and $z = 3$

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$$\begin{aligned} 3x - 2y + 4z &= 1 \\ x + y - 2z &= 3 \\ 2x - 3y + 6z &= 8 \end{aligned}$$

Dividing the first equation by 3 produces

$$\begin{aligned} x - \tfrac{2}{3}y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ x + y - 2z &= 3 \\ 2x - 3y + 6z &= 8 \end{aligned}$$

Subtracting the first equation from the second equation produces

$$\begin{aligned} x - \tfrac{2}{3}y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ \tfrac{5}{3}y - \tfrac{10}{3}z &= \tfrac{8}{3} \\ 2x - 3y + 6z &= 8 \end{aligned}$$

Adding –2 times the first equation to the third equation produces

$$\begin{aligned} x - \tfrac{2}{3}y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ \tfrac{5}{3}y - \tfrac{10}{3}z &= \tfrac{8}{3} \\ -\tfrac{5}{3}y + \tfrac{10}{3}z &= \tfrac{22}{3} \end{aligned}$$

Equations 2 and 3 cannot both be satisfied. So, the original system of equations has no solution

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$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Dividing the first equation by 2 produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Adding –4 times the first equation to the second equation produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ -2x_2 + 8x_3 &= 2 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Adding 2 times the first equation to the third equation produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ -2x_2 + 8x_3 &= 2 \\ 4x_2 - 16x_3 &= -4 \end{aligned}$$

Dividing the second equation by –2 produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ x_2 - 4x_3 &= -1 \\ 4x_2 - 16x_3 &= -4 \end{aligned}$$

Adding −4 times the second equation to the third equation produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ x_2 - 4x_3 &= -1 \\ 0 &= 0 \end{aligned}$$

Choosing $x_3 = t$ as the free variable

The solution is $x_1 = (5/2)t - 1/2$, $x_2 = 4t - 1$, $x_3 = t$, where t is any real number

Determine the value(s) of k such that the system of linear equations has exactly one solution, no solution, infinitely many solutions equations

① $$\begin{aligned} 4x + ky &= 6 \\ kx + y &= -3 \end{aligned}$$

Dividing the first equation by 4 produces $$\begin{aligned} x + \tfrac{k}{4}y &= \tfrac{3}{2} \\ kx + y &= -3 \end{aligned}$$

Adding $-k$ times the first equation to the second equation produces

$$\begin{aligned} x + \tfrac{k}{4}y &= \tfrac{3}{2} \\ (1-\tfrac{k^2}{4})y &= -\tfrac{3}{2}k - 3 \end{aligned}$$

$$(1-\tfrac{k^2}{4}) = 0 \quad \Rightarrow k = \pm 2$$

$k = 2 \Rightarrow$ produces $$\begin{aligned} x + \tfrac{1}{2}y &= \tfrac{3}{2} \\ 0 &= -6 \end{aligned} \quad \Rightarrow \text{No solution}$$

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$k=-2 \Rightarrow$ produces $\begin{array}{rcl} x + \frac{1}{2}y & = & \frac{3}{2} \\ 0 & = & 0 \end{array}$ $\Rightarrow$ Infinitely many solutions

$k \neq \pm 2 \quad \Rightarrow$ exactly one solution

② $\begin{array}{rcl} x + ky & = & 0 \\ kx + y & = & 0 \end{array}$

Adding $-k$ times the first equation to the second equation produces

$$\begin{array}{rcl} x + ky & = & 0 \\ (1-k^2)y & = & 0 \end{array}$$

$$(1-k^2)=0 \quad \Rightarrow k=\pm 1$$

$k=\pm 1$ produces $\begin{array}{rcl} x + y & = & 0 \\ 0 & = & 0 \end{array}$ $\Rightarrow$ Infinitely many solutions

$k \neq \pm 1 \quad \Rightarrow$ exactly one solution (trivial solution $x=y=0$)

③

$$\begin{aligned} x &+ 2y + kz = 6 \\ 3x &+ 6y + 8z = 4 \end{aligned}$$

Reduce the system to row-echelon form

$$\begin{aligned} x + 2y + \quad kz &= 6 \\ (8-3k)z &= -14 \end{aligned}$$

$k = 8/3 \Rightarrow$ no solution

$k \neq 8/3 \Rightarrow$ infinitely many solutions

Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination

①
$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ 2 & 1 & 8 \end{bmatrix} \quad r_{12}^{(-2)} \rightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & -3 & -6 \end{bmatrix} \quad r_{2}^{(-1/3)} \rightarrow \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} x + 2y &= 7 \\ y &= 2 \end{aligned}$$

Using back-substitution you find that $x = 3$ and $y = 2$. Or using Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix} \quad r_{21}^{(-2)} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad \Rightarrow x = 3 \text{ and } y = 2$$

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$$\begin{aligned} -3x &+ 5y = -22 \\ 3x &+ 4y = 4 \\ 4x &- 8y = 32 \end{aligned}$$

$$\begin{bmatrix} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} r_1^{(-1/3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} r_{12}^{(-3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 4 & -8 & 32 \end{bmatrix} r_{13}^{(-4)} \rightarrow$$

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix} r_2^{(1/9)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix} r_{23}^{(4/3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x - (5/3)y &= 22/3 \\ y &= -2 \end{aligned}$$

Using back-substitution you find that $x = 4$ and $y = -2$

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③

$$\begin{aligned} x \quad\quad\quad\quad -3z &= -2 \\ 3x + y - 2z &= 5 \\ 2x + 2y + z &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{bmatrix} r_{12}^{(-3)} \to \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 2 & 2 & 1 & 4 \end{bmatrix} r_{13}^{(-2)} \to \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{bmatrix}$$

$$r_{23}^{(-2)} \to \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{bmatrix} \quad r_{3}^{(-1/7)} \to \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Using back-substitution you find that $x = 4$, $y = -3$ and $z = 2$
Or using Gauss-Jordan elimination

$$\begin{bmatrix}1 & 0 & -3 & -2\\0 & 1 & 7 & 11\\0 & 0 & 1 & 2\end{bmatrix} r_{32}^{(-7)} \to \begin{bmatrix}1 & 0 & -3 & -2\\0 & 1 & 0 & -3\\0 & 0 & 1 & 2\end{bmatrix} r_{31}^{(3)} \to \begin{bmatrix}1 & 0 & 0 & 4\\0 & 1 & 0 & -3\\0 & 0 & 1 & 2\end{bmatrix} \Rightarrow x = 4,\ y = -3 \text{ and } z = 2$$

4

$$\begin{aligned} x + y - 5z &= 3\\ x - 2z &= 1\\ 2x - y - z &= 0 \end{aligned}$$

$$\begin{bmatrix}1 & 1 & -5 & 3\\1 & 0 & -2 & 1\\2 & -1 & -1 & 0\end{bmatrix} r_{12}^{(-1)} \to \begin{bmatrix}1 & 1 & -5 & 3\\0 & -1 & 3 & -2\\2 & -1 & -1 & 0\end{bmatrix} r_{3}^{(-1)} \to \begin{bmatrix}1 & 1 & -5 & 3\\0 & 1 & -3 & 2\\2 & -1 & -1 & 0\end{bmatrix}$$

$$r_{13}^{(-2)} \to \begin{bmatrix}1 & 1 & -5 & 3\\0 & 1 & -3 & 2\\0 & -3 & 9 & -6\end{bmatrix} \quad r_{23}^{(3)} \to \begin{bmatrix}1 & 1 & -5 & 3\\0 & 1 & -3 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\begin{aligned} x + y - 5z &= 3 \\ y - 3z &= 2 \\ 0 &= 0 \end{aligned}$$

Choosing $z = t$ as the free variable

The solution is $x = 1 + 2t$, $y = 2 + 3t$, $z = t$, where t is any real number

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$$\begin{aligned} 2x \qquad + 3z &= 3 \\ 4x - 3y + 7z &= 5 \\ 8x - 9y + 15z &= 10 \end{aligned}$$

$$\begin{bmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} r_1^{(1/2)} \to \begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} r_{12}^{(-4)} \to \begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 0 & -3 & 1 & -1 \\ 8 & -9 & 15 & 10 \end{bmatrix}$$

$$r_{13}^{(-8)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{bmatrix} r_{23}^{(-3)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_{2}^{(-1/3)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the third row corresponds to the equation $0 = 1$, there is no solution to the original system

Graph the system of linear equations. Solve the system and interpret your answer

① $$\begin{aligned} x + 3y &= 2 \\ -x + 2y &= 3 \end{aligned}$$

② $$\begin{aligned} x + 3y &= 17 \\ 4x + 3y &= 7 \end{aligned}$$

③ $$\begin{aligned} \frac{x}{4} + \frac{y}{6} &= 1 \\ x - y &= 3 \end{aligned}$$

Solve the system of linear equations

① $$\begin{aligned} 3u + v &= 240 \\ u + 3v &= 240 \end{aligned}$$

② $$\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$$

③ $$\begin{aligned} x - y - z &= 0 \\ x + 2y - z &= 6 \\ 2x - z &= 5 \end{aligned}$$

④ $$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

⑤ $$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

Determine the value(s) of k such that the system of linear equations has exactly one solution, no solution, infinitely many solutions equations

1. $$\begin{aligned} x + ky &= 2 \\ kx + y &= 4 \end{aligned}$$

2. $$\begin{aligned} 4x + ky &= 0 \\ kx + y &= 0 \end{aligned}$$

3. $$\begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 9z &= -1 \end{aligned}$$

Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination

1. $$\begin{aligned} x + 3y &= 11 \\ 3x + y &= 9 \end{aligned}$$

2. $$\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$$

3. $$\begin{aligned} 2x - 2y + 3z &= 22 \\ 3y - z &= 24 \\ 6x - 7y &= -22 \end{aligned}$$

4. $$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

5. $$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$