

Lecture 2: Matrices

CECC122: Linear Algebra and Matrix Theory

Manara University

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2.1 Operations with Matrices

■ Matrix:

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(K)$$

K : a field (R or C)

$M_{m \times n}(R)$: Real matrices

$M_{m \times n}(C)$: Complex matrices

(i, j) -th entry: a_{ij}

row: m

column: n

size: $m \times n$

■ i -th row vector

$$r_i = [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}] \quad \text{row matrix}$$

Find the 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

$$a_{ij} = i + j$$

$$A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

- **j -th column vector**

$$c_j = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix} \quad \text{column matrix}$$

- **Square matrix:** $m = n$

- **Diagonal matrix:**

$$A = \text{diag}(d_1, d_2, \dots, d_n) = \begin{bmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{bmatrix} \in M_{n \times n}(K)$$

■ **Trace:**

If $A = [a_{ij}]_{n \times n}$ Then $Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^n a_{ii}$

■ **Ex 1:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \Rightarrow r_1 = [1 \ 2 \ 3], \quad r_2 = [4 \ 5 \ 6]$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = [c_1 \ c_2 \ c_3] \Rightarrow c_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

■ **Complex matrices**

$$\begin{bmatrix} 1 & -1 & i \\ 3 & 2i & 0 \end{bmatrix}, \quad [1 \ i \ -i \ 1], \quad \begin{bmatrix} 1 \\ \sqrt{2}i \end{bmatrix}, \quad \begin{bmatrix} 1+i & 1 \\ i & 1-i \end{bmatrix}$$

- **Equal matrix:**

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ Then $A = B$ if and only if $a_{ij} = b_{ij} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n$

- **Ex 2: (Equal matrix)**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If $A = B$ Then $a = 1, b = 2, c = 3, d = 4$

- **Matrix addition:**

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ Then $A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$

- **Ex 3: (Matrix addition)**

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- **Scalar multiplication:**

If $A = [a_{ij}]_{m \times n}$, c : scalar ($\in K$) Then $cA = [ca_{ij}]_{m \times n}$

- **Matrix subtraction:**

$$A - B = A + (-1)B$$

■ **Ex 4: (Scalar multiplication and matrix subtraction)**

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \quad \text{Find (a) } 3A, (b) -B, (c) 3A - B$$

Sol:

$$(a) \quad 3A = 3 \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

$$(b) \quad -B = (-1) \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$$

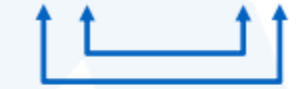
(c)

$$3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$$

■ **Matrix multiplication:**

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$

Then $AB = [a_{ij}]_{m \times n} [b_{ij}]_{n \times p} = [c_{ij}]_{m \times p}$



Size of AB

where $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & \dots & b_{2j} & \dots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1j} & \dots & c_{1p} \\ \vdots & & \vdots & & \vdots \\ c_{i1} & \dots & c_{ij} & \dots & c_{ip} \\ \vdots & & \vdots & & \vdots \\ c_{m1} & \dots & c_{mj} & \dots & c_{mp} \end{bmatrix}$$

- **Notes:** (1) $A + B = B + A$, (2) $AB \neq BA$
- **Ex 5: (Find AB)**

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

Sol: $AB = \begin{bmatrix} (-1)(-3) + (3)(-4) & (-1)(2) + (3)(1) \\ (4)(-3) + (-2)(-4) & (4)(2) + (-2)(1) \\ (5)(-3) + (0)(-4) & (5)(2) + (0)(1) \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$

- **Zero matrix:** $O_{m \times n}$
- **Identity matrix of order n :** I_n
- **Properties of matrix addition and scalar multiplication:**

If $A, B, C \in M_{m \times n}(K)$ then

$$(1) A + B = B + A$$

$$(2) A + (B + C) = (A + B) + C$$

$$(3) (cd)A = c(dA)$$

$$(4) 1A = A$$

$$(5) c(A + B) = cA + cB$$

$$(6) (c + d)A = cA + dA$$

- **Properties of zero matrices:**

If $A \in M_{m \times n}(K)$, c : scalar then

$$(1) A + O_{m \times n} = A$$

$$(2) A + (-A) = O_{m \times n}$$

$$(3) cA = O_{m \times n} \Rightarrow c = 0 \text{ or } A = O_{m \times n}$$

■ Notes:

- (1) $O_{m \times n}$: the additive identity for the set of all $m \times n$ matrices
- (2) $-A$: the additive inverse of A

■ Properties of matrix multiplication:

- (1) $A(BC) = (AB)C$
- (2) $A(B + C) = AB + AC$
- (3) $(A + B)C = AC + BC$
- (4) $c(AB) = (cA)B = A(cB)$

■ Properties of identity matrix:

If $A \in M_{m \times n}(K)$, then

- (1) $AI_n = A$
- (2) $I_m A = A$

Matrix Multiplication by Columns and by Rows

Sometimes it may be desirable to find a particular row or column of a matrix product AB without computing the entire product. The following results, whose proofs are left as exercises, are useful for that purpose:

$$j\text{th column matrix of } AB = A[j\text{th column matrix of } B]$$

EX

$$i\text{th row matrix of } AB = [i\text{th row matrix of } A]B$$

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

$\uparrow$ Second column of B $\uparrow$ Second column of AB

$$\text{First row of } A \rightarrow [1 \ 2 \ 4] \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = [12 \ 27 \ 30 \ 13] \leftarrow \text{First row of } AB$$

■ **Transpose of a matrix:**

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \in M_{m \times n}(K)$$

$$\text{Then } A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \in M_{n \times m}(K)$$

■ Ex 1: (Find the transpose of the following matrix)

$$(a) A = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \quad (b) A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad (c) A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}$$

Sol: (a) $A = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 2 & 8 \end{bmatrix}$

$$(b) A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}$$

- **Properties of transposes:**

(1) $(A^T)^T = A$

(2) $(A + B)^T = A^T + B^T$

(3) $(cA)^T = c(A^T)$

(4) $(AB)^T = B^T A^T$

- **Symmetric matrix:**

A square matrix A is **symmetric** if $A^T = A$

- **Skew-symmetric matrix:**

A square matrix A is **skew-symmetric** if $A^T = -A$

■ **Ex 2:**

If $A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}$ is symmetric, find a, b, c ?

Sol:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ a & 4 & 5 \\ b & c & 6 \end{bmatrix}, A^T = \begin{bmatrix} 1 & a & b \\ 2 & 4 & c \\ 3 & 5 & 6 \end{bmatrix} \quad A = A^T \Rightarrow a = 2, b = 3, c = 5$$

■ **Ex 3:**

If $A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric, find a, b, c ?

Sol:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 3 \\ b & c & 0 \end{bmatrix}, \quad -A^T = \begin{bmatrix} 0 & -a & -b \\ -1 & 0 & -c \\ -2 & -3 & 0 \end{bmatrix}$$

$$A = -A^T \Rightarrow a = -1, \quad b = -2, \quad c = -3$$

■ **Notes:**

(1) AA^T is symmetric

(2) Every square matrix $A \in M_n(R)$ can be expressed as the sum of a symmetric matrix B and a skew-symmetric matrix C

$$B = \frac{1}{2}(A + A^T), \quad C = \frac{1}{2}(A - A^T)$$

■ **Ex 4:**

Show that AB and BA are not equal for the matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

■ **Note:** $AB \neq BA$

- **Real (Complex) number:**

$$ac = bc, c \neq 0 \Rightarrow a = b \quad \text{(Cancellation law)}$$

- **Matrix:**

$$AC = BC, \quad C \neq O$$

(1) If C is invertible, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- **Ex 5: (An example in which cancellation is not valid)**

Show that $AC = BC$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

Sol:

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \quad BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

So $AC = BC$ but $A \neq B$

2.3 The Inverse of a Matrix

- **Inverse matrix:**

Consider $A \in M_{n \times n}(K)$

If there exists a matrix $B \in M_{n \times n}(K)$ such that $AB = BA = I_n$,

Then (1) A is **invertible** (or **nonsingular**)

(2) B is **the inverse** of A

- **Note:**

A matrix that does not have an inverse is called **noninvertible** (or **singular**).

- **Theorem 2.1 (The inverse of a matrix is unique):**

If B and C are both inverses of the matrix A , then $B = C$.

Consequently, the inverse of a matrix is unique.

- **Notes:**

(1) The inverse of A is denoted by A^{-1}

(2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$\left[A \mid I \right] \longrightarrow \left[I \mid A^{-1} \right] \quad \text{Gauss-Jordan Elimination}$$

- Ex 2: (Find the inverse of the following matrix)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Sol: $\left[A \mid I \right] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ -6 & 2 & 3 & \vdots & 0 & 0 & 1 \end{array} \right]$

$$\xrightarrow{r_{12}^{(-1)}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ -6 & 2 & 3 & \vdots & 0 & 0 & 1 \end{array} \right] \xrightarrow{r_{13}^{(6)}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & -4 & 3 & \vdots & 6 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{r_{23}^{(4)}} \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 0 & -1 & \vdots & 2 & 4 & 1 \end{bmatrix} \xrightarrow{r_3^{(-1)}} \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -2 & -4 & -1 \end{bmatrix}$$

$$\xrightarrow{r_{32}^{(1)}} \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & -3 & -3 & -1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & -1 \end{bmatrix} \xrightarrow{r_{21}^{(1)}} \begin{bmatrix} 1 & 0 & 0 & \vdots & -2 & -3 & -1 \\ 0 & 1 & 0 & \vdots & -3 & -3 & -1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & -1 \end{bmatrix} = \left[I : A^{-1} \right]$$

So the matrix A is invertible, and its inverse is $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$

■ **Check:** $AA^{-1} = A^{-1}A = I$