Lecture 3: Matrices

CECC122: Linear Algebra and Matrix Theory

Manara University

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- **Power of a square matrix:**

$$(1)\ A^0 = I$$

$$(2)\ A^k = \underbrace{A A \cdots A}_{k \text{ factors}} \qquad (k > 0)$$

$$(3)\ A^r \cdot A^s = A^{r+s} \qquad r, s: \text{integers}$$

$$(4)\ (A^r)^s = A^{rs}$$

$$(5)\ D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- **Theorem 2.2 (Properties of inverse matrices):**

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(2) A^{k} is invertible and $(A^{k})^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^{k} = A^{-k}$

(3) cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1},\ c \neq 0$

(4) A^{T} is invertible and $(A^{T})^{-1} = (A^{-1})^{T}$

- **Theorem 2.3 (The inverse of a product):**

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

- **Note:**

$$(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- **Theorem 2.5 (Systems of equations with unique solutions):**

If A is an invertible matrix, then the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution given by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

- Note:

For square systems (those having the same number of equations as variables), Theorem 2.5 can be used to determine whether the system has a unique solution.

- Note: $A\mathbf{x}=\mathbf{b}$ (A is an invertible matrix)

$$\left[A \mid \mathbf{b}\right] \xrightarrow{A^{-1}} \left[A^{-1}A \mid A^{-1}\mathbf{b}\right] = \left[I \mid A^{-1}\mathbf{b}\right]$$

- Ex 3: Use an inverse matrix to solve each system

$$(a)\quad \begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$(a)\quad \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$(b)\quad \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2.4 Elementary Matrices

- Row elementary matrix:

  An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

(1) $R_{ij} = r_{ij}(I)$	Interchange two rows.
(2) $R_i^{(k)} = r_i^{(k)}(I) \quad (k \neq 0)$	Multiply a row by a nonzero constant.
(3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$	Add a multiple of a row to another row.

- Note:

  Only do a single elementary row operation.

- Ex 1: (Elementary matrices and non elementary matrices)

$(a)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Yes $(r_2^{(3)}(I_3))$

$(b)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

No (not square)

$(c)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

No (Row multiplication must be by a nonzero constant)

$(d)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Yes $(r_{23}(I_3))$

$(e)\ \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Yes $(r_{12}^{(2)}(I_2))$

$(f)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

No (Use two elementary row operations)

- **Theorem 2.6 (Representing elementary row operations):**

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$

- **Notes:**

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

- Ex 2: (Elementary matrices and elementary row operation)

$$(a)\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 1 & -3 & 6 \\ 3 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ 0 & 2 & 1 \\ 3 & 2 & -1 \end{bmatrix} \quad (r_{12}(A) = R_{12}A)$$

$$(b)\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 2 & 6 & -4 \\ 0 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & 1 \end{bmatrix} \quad (r_2^{(\frac{1}{2})}(A) = R_2^{(\frac{1}{2})}A)$$

$$(c)\ \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -2 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 4 & 5 \end{bmatrix} \quad (r_{12}^{(2)}(A) = R_{12}^{(2)}A)$$

- Ex 3: (Using elementary matrices)

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A=\begin{bmatrix}0 & 1 & 3 & 5\\ 1 & -3 & 0 & 2\\ 2 & -6 & 2 & 0\end{bmatrix}$$

Sol:

$$A_1=r_{12}(A)=E_1A=\begin{bmatrix}1 & -3 & 0 & 2\\ 0 & 1 & 3 & 5\\ 2 & -6 & 2 & 0\end{bmatrix}$$

$$E_1=r_{12}(I_3)=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A_2=r_{13}^{(-2)}(A_1)=E_2A_1=\begin{bmatrix}1 & -3 & 0 & 2\\ 0 & 1 & 3 & 5\\ 0 & 0 & 2 & -4\end{bmatrix}$$

$$E_2=r_{13}^{(-2)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3A_2 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$B = E_3E_2E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- **Row-equivalent:**

Matrix B is row-equivalent to A if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- **Theorem 2.7: (Elementary matrices are invertible)**

If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- **Notes:**

(1) $(R_{ij})^{-1} = R_{ij}$

(2) $(R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$

(3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

- Ex 4:

Elementary Matrix	Inverse Matrix
$E_1=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=R_{12}$	$(R_{12})^{-1}=E_1^{-1}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=R_{12}$ (Elementary Matrix)
$E_2=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}=R_{13}^{(-2)}$	$(R_{13}^{(-2)})^{-1}=E_2^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}=R_{13}^{(2)}$ (Elementary Matrix)
$E_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{bmatrix}=R_3^{(\frac{1}{2})}$	$(R_3^{(\frac{1}{2})})^{-1}=E_3^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}=R_3^{(2)}$ (Elementary Matrix)

- **Theorem 2.8 (A property of invertible matrices):**

A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

- **Ex 5:** Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_1^{(-1)}} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-3)}} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2^{(1/2)}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore $R_{21}^{(-2)} R_2^{(\frac{1}{2})} R_{12}^{(-3)} R_1^{(-1)} A = I$

Thus $A = (R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$

$$= R_1^{(-1)}R_{12}^{(3)}R_2^{(2)}R_{21}^{(2)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- **Note:**

If A is invertible then

$$E_k \cdots E_3E_2E_1A = I$$

$$A^{-1} = E_k \cdots E_3E_2E_1$$

$$A = E_1^{-1}E_2^{-1}E_3^{-1} \cdots E_k^{-1}$$

- **Theorem 2.9 (Equivalent conditions):**

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \boldsymbol{b}$ has a unique solution for every $n \times 1$ column matrix $\boldsymbol{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n.

(5) A can be written as the product of elementary matrices.

- *LU*–factorization:

If the $n \times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an LU–factorization of A

- Note:

If a square matrix A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is easy to find an LU–factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU \quad (L = E_1^{-1} E_2^{-1} \cdots E_k^{-1})$$

- **Ex 6:(*LU*–factorization)**

$$(a)\ A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \qquad (b)\ A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}\xrightarrow{r_{12}^{(-1)}}\begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix}=U$$

$$\Rightarrow R_{12}^{(-1)}A=U$$

$$\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&-3&0\\0&1&3\\2&-10&2\end{bmatrix}\xrightarrow{r_{13}^{(-2)}}\begin{bmatrix}1&-3&0\\0&1&3\\0&-4&2\end{bmatrix}\xrightarrow{r_{23}^{(4)}}\begin{bmatrix}1&-3&0\\0&1&3\\0&0&14\end{bmatrix}=U$$

$$\Rightarrow R_{23}^{(4)}R_{13}^{(-2)}A=U$$

$$\Rightarrow A=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}U=LU$$

$$\Rightarrow L=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}=R_{13}^{(2)}R_{23}^{(-4)}$$

$$=\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-4&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\2&-4&1\end{bmatrix}$$

- Solving $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ with an LU–factorization of A

$$A\mathbf{x} = \mathbf{b} \quad \text{If } A = LU, \text{ then } LU\mathbf{x} = \mathbf{b}$$

$$\text{Let } \mathbf{y} = U\mathbf{x}, \text{ then } L\mathbf{y} = \mathbf{b}$$

- Two steps:

(1) Write $\mathbf{y} = U\mathbf{x}$, and solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$

(2) Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$

- **Ex 7: (Solving a linear system using LU–factorization)**

$$\begin{aligned} x_1 - 3x_2 \quad &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $\mathbf{y} = U\mathbf{x}$, and solve for $L\mathbf{y} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20 - 2y_1 + 4y_2 = -14 \end{aligned}$$

(2) Solve the following system $U\boldsymbol{x} = \boldsymbol{y}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So $x_3 = -1$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$