Lecture 8: Linear Transformations

CECC122: Linear Algebra and Matrix Theory

Manara University

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6.1 Introduction to Linear Transformations

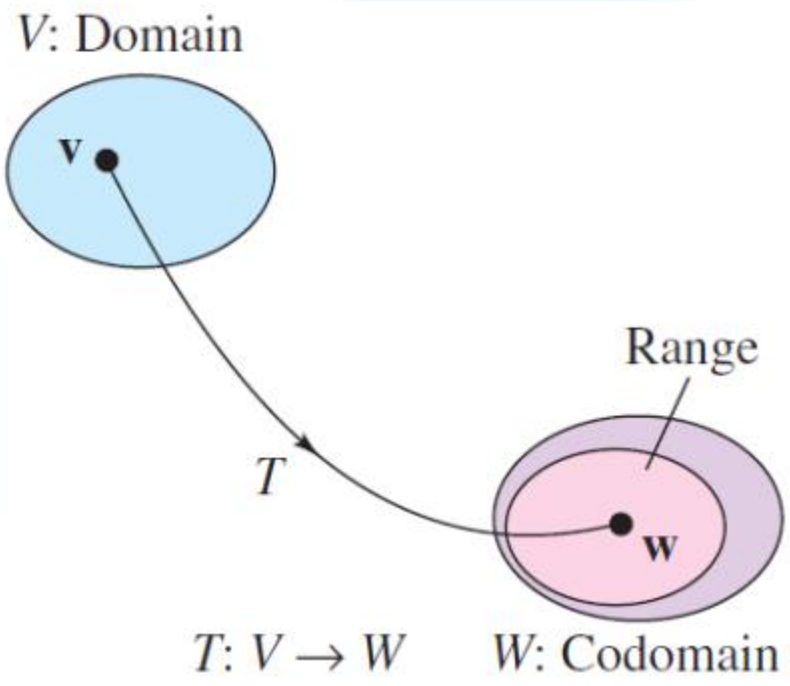
- **Images And Preimages of Functions:**

Function T that maps a vector space V into a vector space W

$$T: V \xrightarrow{\text{Mapping}} W, \qquad V, W: \text{vector spaces}$$

V: the domain of T

W: the codomain of T



- **Image of v under T:**

If v is in V and w is in W such that: $T(v) = w$

Then w is called the image of v under T

- Images And Preimages of Functions:

- The range of T: The set of all images of vectors in V.

- The preimage of $\mathbf{w}$: The set of all $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$.

- Ex 1: (A function from R^2 into R^2)

$$T: R^2 \rightarrow R^2 \qquad \mathbf{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of $\mathbf{v} = (-1, 2)$. (b) Find the preimage of $\mathbf{w} = (-1, 11)$

Sol:

$$(a)\ \mathbf{v} = (-1, 2) \Rightarrow T(\mathbf{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$$

$$(b)\ T(\mathbf{v}) = \mathbf{w} = (-1, 11) \Rightarrow T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2) = (-1, 11)$$

$$\Rightarrow v_1 - v_2 = -1$$
$$v_1 + 2v_2 = 11$$
$$\Rightarrow v_1 = 3, \ v_2 = 4$$

Thus {(3, 4)} is the preimage of $\mathbf{w} = (-1, 11)$.

- **Linear Transformation (L.T.):**

V, W: vector spaces

$T: V \to W$: Linear Transformation

$$(1)\ T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V$$

$$(2)\ T(c\mathbf{u}) = cT(\mathbf{u}), \quad \forall c \in R$$

- Ex 2: (Functions that are not linear transformations)

$(a)\ f(x) = \sin x$

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

$$\sin(\frac{\pi}{2} + \frac{\pi}{3}) \neq \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{3})$$

not a linear transformations

$(b)\ f(x) = x^2$

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2 \qquad (1+2)^2 \neq 1^2 + 2^2$$

not a linear transformations

$(c)\ f(x) = x + 1$

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

not a linear transformations

- Notes: Two uses of the term "linear"

(1) $f(x) = x + 1$ is called a linear function because its graph is a line.

(2) $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication

- Zero transformation:

$$T: V \to W \qquad T(\mathbf{v}) = \mathbf{0}, \ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v}, \ \forall \mathbf{v} \in V$$

- **Theorem 6.1: (Properties of linear transformations)**

$$T: V \to W, \quad \boldsymbol{u}, \boldsymbol{v} \in V$$

$$(1)\ T(\mathbf{0}) = \mathbf{0}$$

$$(2)\ T(-\boldsymbol{v}) = -T(\boldsymbol{v})$$

$$(3)\ T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$$

$$(4)\ \text{If } \boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n \text{ then}$$

$$T(\boldsymbol{v}) = T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n)$$

$$= c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_nT(\boldsymbol{v}_n)$$

- Ex 3: (Linear transformations and bases)

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4), \quad T(0,1,0) = (1,5,-2), \quad T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$\begin{aligned} T(2,3,-2) &= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \\ &= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1) \\ &= (7,7,0) \end{aligned}$$

6.2 The Kernel and Range of a Linear Transformation

- **Kernel of a linear transformation T:**

  Let $T: V \to W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the kernel of T and is denoted by ker(T).

$$\ker(T) = \{\mathbf{v} \mid T(\mathbf{v}) = \mathbf{0}, \forall \mathbf{v} \in V\}$$

- **Ex 1: (The kernel of the zero and identity transformations)**

  (a) $T(\mathbf{v}) = \mathbf{0}$ (the zero transformation $T: V \to W$)

$$\ker(T) = V$$

  (b) $T(\mathbf{v}) = \mathbf{v}$ (the identity transformation $T: V \to V$)

$$\ker(T) = \{\mathbf{0}\}$$

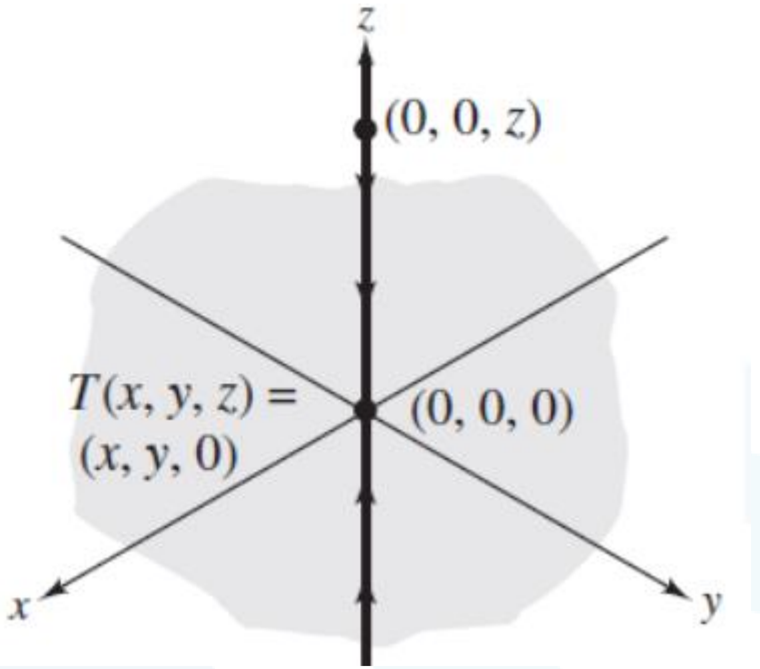
- **Ex 2: (Finding the kernel of a L.T.)**

$T(\mathbf{v}) = (x, y, 0)\; T: R^3 \rightarrow R^3$

$\ker(T) = ?$

Sol:

$\ker(T) = \{(0, 0, z) \mid z \text{ is a real number}\}$



- **Ex 3: (Finding the kernel of a linear transformation)**

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T: R^3 \rightarrow R^2)$$

$\ker(T) = ?$

Sol:

$$\ker(T) = \{(x_1,x_2,x_3) \mid T(x_1,x_2,x_3) = (0,0),\ \boldsymbol{x} = (x_1,x_2,x_3) \in R^3\}$$

$$T(x_1,x_2,x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \{t(1,-1,1) \mid t \text{ is a real number}\} = \operatorname{span}\{(1,-1,1)\}$$

- **Theorem 6.3: (The kernel is a subspace of V)**

  The kernel of a linear transformation $T: V \rightarrow W$ is a subspace of the domain V.

- **Range of a linear transformation T:**

  Let $T: V \rightarrow W$ be a L.T.

  Then the set of all vectors $\mathbf{w}$ in W that are images of vectors in V is called the range of T and is denoted by range(T)

$$\text{range}(T) = \{T(\mathbf{v}) \mid \forall\, \mathbf{v} \in V\}$$

- **Theorem 6.4: (The range of T is a subspace of W)**

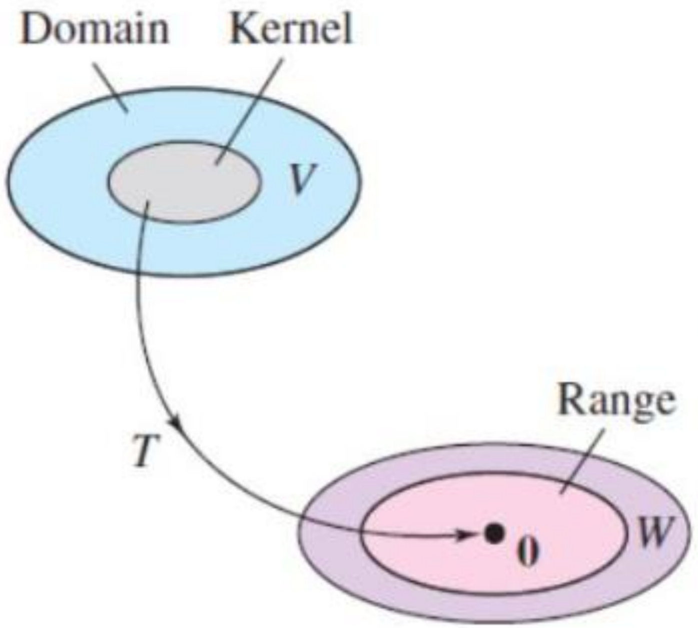
  The range of a linear transformation $T: V \rightarrow W$ is a subspace of the W

- Notes:

  $T: V \to W$: is Linear Transformation

  (1) ker(T) is a subspace of V

  (2) Range(T) is a subspace of W



- Rank of a linear transformation $T: V \to W$:

  rank(T) = the dimension of the range of T

- Nullity of a linear transformation $T: V \to W$:

  nullity(T) = the dimension of the kernel of T

- Note:

  Let $T: R^n \to R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$. Then

  $\Rightarrow$ rank(T) = rank(A), nullity(T) = nullity(A)

- **Theorem 6.5: (Sum of rank and nullity)**

Let $T: V \rightarrow W$ be a L.T. from an n-dimensional vector space V into a vector space W. Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$$

- **Ex 4: (Finding rank and nullity of a linear transformation)**

Find the rank and nullity of the L.T. $T: R^3 \rightarrow R^3$ defined by

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$$\text{rank}(T) = \text{rank}(A) = 2$$

$$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

- Ex 5: (Finding rank and nullity of a linear transformation)

Let $T: R^5 \rightarrow R^7$ be a linear transformation

(a) Find the dimension of the kernel of T if the dimension of the range is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $\ker(T) = \{\mathbf{0}\}$

Sol:

(a) $\dim(\text{domain of } T) = 5$

$\dim(\ker \text{ of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$

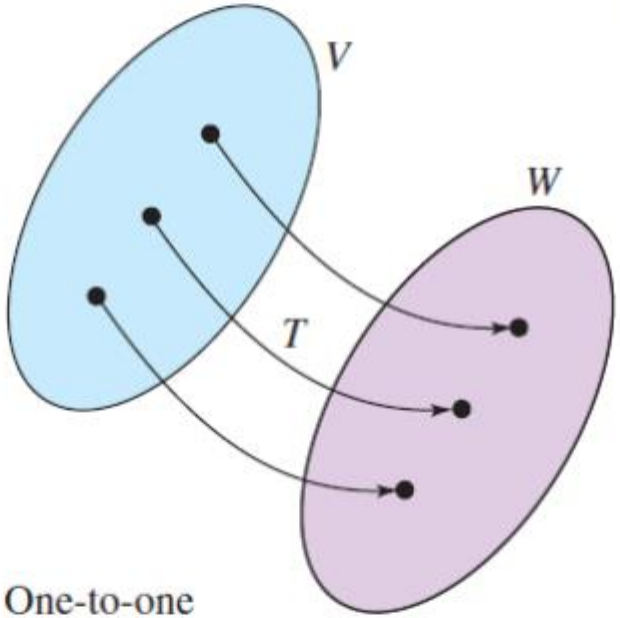
(b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

(c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

- One-to-one:

A function $T\colon V \to W$ is one-to-one when the preimage of every $\mathbf{w}$ in the range consists of a single vector

T is one-to-one if and only if, for all $\mathbf{u}$ and $\mathbf{v}$ in V, $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.



One-to-one   Not one-to-one

- **Onto:**

A function $T: V \rightarrow W$ is onto when every element in W has a preimage in V. (T is onto W when W is equal to the range of T)

- **Theorem 6.6: (One-to-one linear transformation)**

Let $T: V \rightarrow W$ be a linear transformation. Then T is one-to-one iff $\ker(T) = \{\mathbf{0}\}$

- **Ex 6: (One-to-one and not one-to-one linear transformation)**

(a) The linear transformation $T: M_{3x2} \rightarrow M_{2x3}$ given by $T(A) = A^T$ is one-to-one because its kernel consists of only the mxn zero matrix

(b) The zero transformation $T: R^3 \rightarrow R^3$ is not one-to-one because its kernel is all of R^3

- **Theorem 6.7: (Onto linear transformation)**

  Let $T: V \rightarrow W$ be a linear transformation, where W is finite dimensional Then T is onto iff the rank of T is equal to the dimension of W.

- **Theorem 6.8: (One-to-one and onto linear transformation)**

  Let $T: V \rightarrow W$ be a linear transformation, with vector space V and W both of dimension n. Then T is one-to-one iff it is onto.

- **Ex 7:**

Let $T: R^n \rightarrow R^m$ be a L.T. given by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither

$$(a)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ (b)\ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ (c)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix},\ (d)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$T: R^n \rightarrow R^m$	dim(domain of T)	rank(T)	nullity(T)	one-to-one	onto
(a) $T: R^3 \rightarrow R^3$	3	3	0	Yes	Yes
(b) $T: R^2 \rightarrow R^3$	2	2	0	Yes	No
(c) $T: R^3 \rightarrow R^2$	3	2	1	No	Yes
(d) $T: R^3 \rightarrow R^3$	3	2	1	No	No

6.3 Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \rightarrow R^3$

$$(1)\ T(x_1,x_2,x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)\ T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
  - It is simpler to write.
  - It is simpler to read.
  - It is more easily adapted for computer use.

- **Theorem 6.9: (Standard matrix for a linear transformation)**

Let $T: R^n \rightarrow R^m$ be a linear transformation such that

$$T(\mathbf{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}, \quad T(\mathbf{e} \quad) = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \cdots, \quad T(\mathbf{e} \quad) = \begin{bmatrix} a_{1n} \\ a_{2n} \end{bmatrix},$$

[redacted]

$$\begin{bmatrix} a_{m1} \end{bmatrix} \quad \begin{bmatrix} a_{m2} \end{bmatrix} \quad \begin{bmatrix} a_{mn} \end{bmatrix}$$

then the mxn matrix whose n columns correspond to $T(\mathbf{e}_i)$

$$A = \left[T(\mathbf{e}_1) \,|\, T(\mathbf{e}_2) \,|\, \cdots \,|\, T(\mathbf{e}_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v}$ in R^n. A is called the standard matrix for T

- **Ex 1: (Finding the standard matrix of a linear transformation)**

Find the standard matrix for the L.T. $T: R^3 \rightarrow R^2$ defined by $T(x,y,z) = (x - 2y,\ 2x + y)$

Sol:

Vector Notation	Matrix Notation
$T(\mathbf{e}_1) = T(1,\ 0,\ 0) = (1,\ 2)$	$T(\mathbf{e}_1) = T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}1\\2\end{bmatrix}$
$T(\mathbf{e}_2) = T(0,\ 1,\ 0) = (-2,\ 1)$	$T(\mathbf{e}_2) = T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}-2\\1\end{bmatrix}$
$T(\mathbf{e}_3) = T(0,\ 0,\ 1) = (0,\ 0)$	$T(e_3) = T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}0\\0\end{bmatrix}$

$$A = [T(e_1) \mid T(e_2) \mid T(e_3)] = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

- **Check:**

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ 2x+y \end{bmatrix}$$

i.e. $T(x,y,z) = (x-2y, 2x+y)$

- **Notes:**

(1) The standard matrix for the zero transformation from R^n into R^m is the $m \times n$ zero matrix.

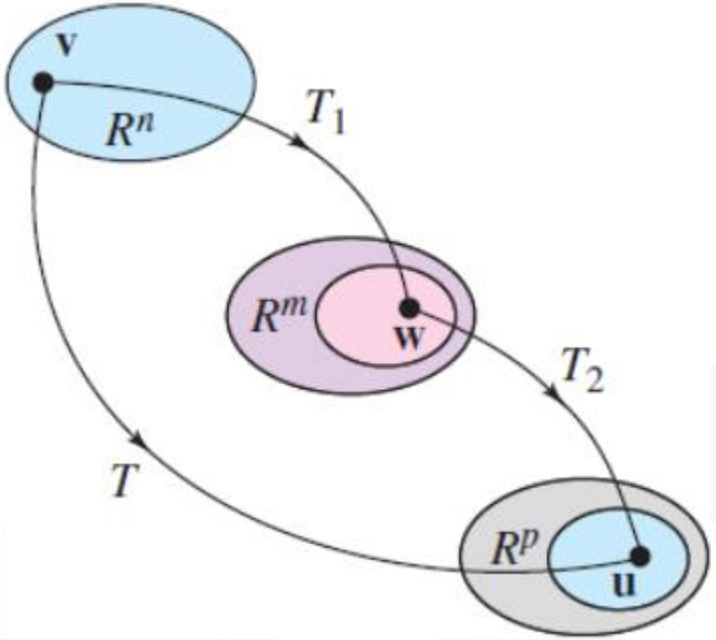
(2) The standard matrix for the identity transformation from R^n into R^n is the $n \times n$ identity matrix I_n

- Composition of $T_1: R^n \rightarrow R^m$ with $T_2: R^m \rightarrow R^p$:

$$T(\mathbf{v}) = T_2(T_1(\mathbf{v})), \quad \mathbf{v} \in R^n$$

$$T = T_2 \circ T_1,$$

domain of T = domain of T_1



- Note: $T_1 \circ T_2 \neq T_2 \circ T_1$

- Theorem 6.10: (Composition of linear transformations)

Let $T_1: R^n \rightarrow R^m$ and $T_2: R^m \rightarrow R^p$ be L.T. with standard matrices A_1 and A_2, then

(1) The composition $T: R^n \rightarrow R^p$, defined by $T(\mathbf{v}) = T_2(T_1(\mathbf{v}))$, is a L. T.

(2) The standard matrix A for T is given the matrix product $A = A_2 A_1$

- Ex 2: (The standard matrix of a composition)

Let T_1 and T_2 be L. T. from R^3 into R^3 such that

$$T_1(x,y,z) = (2x + y,\ 0,\ x + z), \quad T_2(x,y,z) = (x - y,\ z,\ y)$$

Find the standard matrices for the compositions

$$T = T_2 \circ T_1 \text{ and } T' = T_1 \circ T_2$$

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

standard matrices for T_1 standard matrices for T_2

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2 A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1 A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- **Inverse linear transformation:**

If $T_1: R^n \to R^n$ and $T_2: R^n \to R^n$ are L.T. such that for every $\mathbf{v}$ in R^n

$$T_2(T_1(\mathbf{v})) = \mathbf{v} \quad \text{and} \quad T_1(T_2(\mathbf{v})) = \mathbf{v}$$

Then T_2 is called the inverse of T_1 and T_1 is said to be invertible

- **Note:**

If the transformation T is invertible, then the inverse is unique and denoted by T^{-1}.

- **Theorem 6.11: (Existence of an inverse transformation)**

  Let $T: R^n \rightarrow R^n$ be a L.T. with standard matrices, then the following conditions are equivalent

  (1) T is invertible.

  (2) A is invertible.

- **Note:**

  If T is invertible with standard matrix A, then the standard matrix for T^{-1} is A^{-1}.

- **Ex 3: (Finding the inverse of a linear transformation)**

The L. T. $T: R^3 \to R^3$ defined by

$$T(x_1, x_2, x_3) = (2x_1 + 3x_2 + x_3,\ 3x_1 + 3x_2 + x_3,\ 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse.

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$\left[A \mid I_3\right] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{G.J. Elimination}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right] = \left[I \mid A^{-1} \right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\mathbf{v}) = A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

In other words $T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3)$