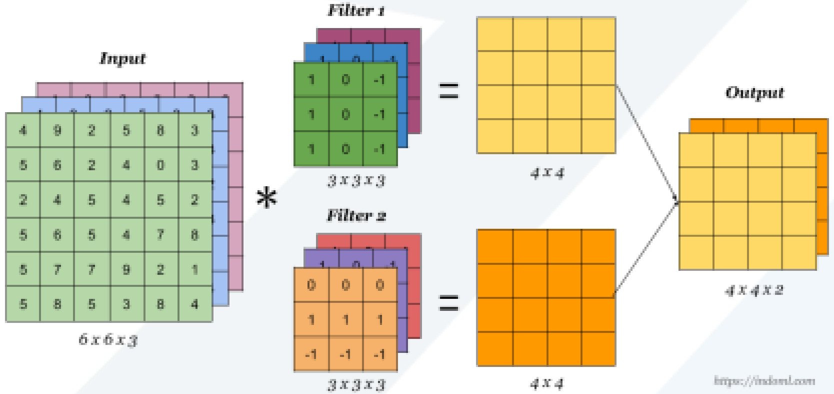
CECC102, CECC122 & CEDC102 : Linear Algebra (and Matrix Theory)

Exercises 1 & 2: Linear Equations and Matrices



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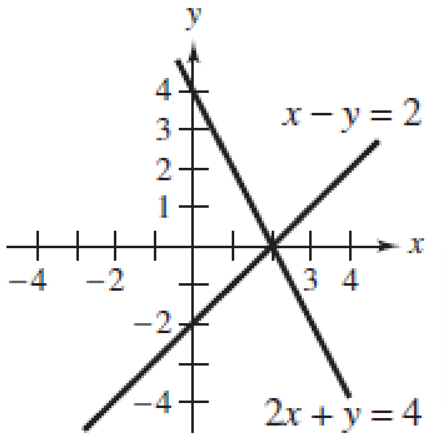
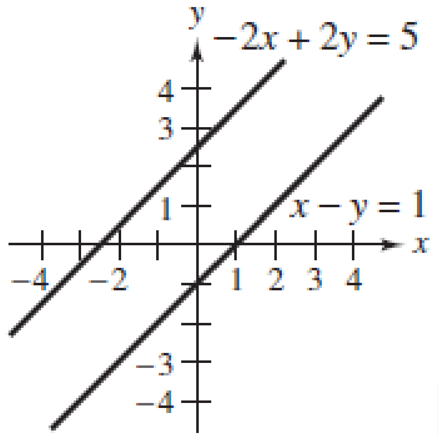
Graph the system of linear equations. Solve the system and interpret your answer

①
$$\begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$$



Adding the first equation to the second produces a new equation, $3x = 6$, or $x = 2$. So, $y = 0$, and the solution is $x = 2$, $y = 0$.

②
$$\begin{aligned} x - y &= 1 \\ -2x + 2y &= 5 \end{aligned}$$



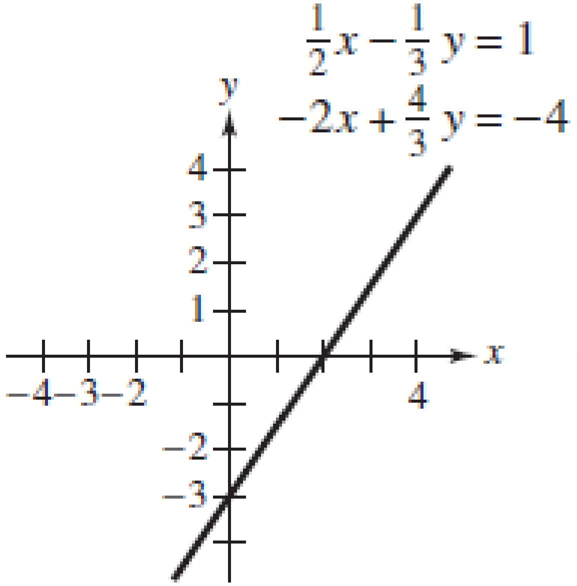
Adding 2 times the first equation to the second produces

$$\begin{aligned} x - y &= 1 \\ 0 &= 7 \end{aligned}$$

The second equation is a false statement, therefore the original system has no solution. The two lines are parallel

③

$$\begin{aligned} \tfrac{1}{2}x - \tfrac{1}{3}y &= 1 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$$



Multiplying the first equation by 2 produces

$$\begin{aligned} x - \tfrac{2}{3}y &= 2 \\ -2x + \tfrac{4}{3}y &= -4 \end{aligned}$$

Adding 2 times the first equation to the second equation produces

$$\begin{aligned} x - \tfrac{2}{3}y &= 2 \\ 0 &= 0 \end{aligned}$$

Choosing $y = t$ as the free variable, $x = (2/3)t + 2$.

So, the solution set is $x = (2/3)t + 2$ and $y = t$, where t is any real number.

Solve the system of linear equations

①
$$\begin{aligned} x_1 - x_2 &= 0 \\ 3x_1 - 2x_2 &= -1 \end{aligned}$$

Adding –3 times the first equation to the second equation produces

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_2 &= -1 \end{aligned}$$

Using back-substitution you can conclude that the system has exactly one solution: $x_1 = -1$and $x_2 = -1$

②
$$\begin{aligned} 3x + 2y &= 2 \\ 6x + 4y &= 14 \end{aligned}$$

Adding –2 times the first equation to the second equation produces

$$\begin{aligned} 3x + 2y &= 2 \\ 0 &= 10 \end{aligned}$$

Because the second equation is a false statement, the original system of equations has no solution.

③

$$\begin{aligned} \tfrac{2}{3}x + \tfrac{1}{6}y &= 0 \\ 4x + y &= 0 \end{aligned}$$

Multiplying the first equation by 3/2 produces

$$\begin{aligned} x + \tfrac{1}{4}y &= 0 \\ 4x + y &= 0 \end{aligned}$$

Adding –4 times the first equation to the second produces

$$\begin{aligned} x + \tfrac{1}{4}y &= 0 \\ 0 &= 0 \end{aligned}$$

Choosing $x = t$ as the free variable, $y = -(1/4)t$

So, the solution set is $x = t$ and $y = -(1/4)t$, where t is any real number

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 6 \\ 2x & - & y & + & z & = & 3 \\ 3x & & & - & z & = & 0 \end{array} \quad (4)$$

Adding –2 times the first equation to the second produces

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ & & -3y & - & z & = & -9 \\ & & -3y & - & 4z & = & -18 \end{array}$$

Dividing the second equation by –3 produces

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ & & y & + & \frac{1}{3}z & = & 3 \\ & & -3y & - & 4z & = & -18 \end{array}$$

Adding 3 times the second equation to the third equation produces

$$\begin{array}{rcrcrcr} x & + & y & + & z & = & 6 \\ & & y & + & \frac{1}{3}z & = & 3 \\ & & & - & 3z & = & -9 \end{array}$$

Dividing the third equation by –3 produces

$$\begin{aligned} x + y + z &= 6 \\ y + \tfrac{1}{3}z &= 3 \\ z &= 3 \end{aligned}$$

Using back-substitution you can conclude that the system has exactly one solution: $x = 1$, $y = 2$, and $z = 3$

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$$\begin{aligned} 3x - 2y + 4z &= 1 \\ x + y - 2z &= 3 \\ 2x - 3y + 6z &= 8 \end{aligned}$$

Dividing the first equation by 3 produces

$$\begin{aligned} x - \tfrac{2}{3}y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ x + y - 2z &= 3 \\ 2x - 3y + 6z &= 8 \end{aligned}$$

Subtracting the first equation from the second equation produces

$$\begin{aligned} x - \tfrac{2}{3}y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ \tfrac{5}{3}y - \tfrac{10}{3}z &= \tfrac{8}{3} \\ 2x - 3y + 6z &= 8 \end{aligned}$$

Adding –2 times the first equation to the third equation produces

$$\begin{aligned} x - \tfrac{2}{3}y + \tfrac{4}{3}z &= \tfrac{1}{3} \\ \tfrac{5}{3}y - \tfrac{10}{3}z &= \tfrac{8}{3} \\ -\tfrac{5}{3}y + \tfrac{10}{3}z &= \tfrac{22}{3} \end{aligned}$$

Equations 2 and 3 cannot both be satisfied. So, the original system of equations has no solution

⑥

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Dividing the first equation by 2 produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ 4x_1 \qquad\quad + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Adding –4 times the first equation to the second equation produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ -2x_2 + 8x_3 &= 2 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

Adding 2 times the first equation to the third equation produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ -2x_2 + 8x_3 &= 2 \\ 4x_2 - 16x_3 &= -4 \end{aligned}$$

Dividing the second equation by –2 produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ x_2 - 4x_3 &= -1 \\ 4x_2 - 16x_3 &= -4 \end{aligned}$$

Adding −4 times the second equation to the third equation produces

$$\begin{aligned} x_1 + \tfrac{1}{2}x_2 - \tfrac{3}{2}x_3 &= 2 \\ x_2 - 4x_3 &= -1 \\ 0 &= 0 \end{aligned}$$

Choosing $x_3 = t$ as the free variable

The solution is $x_1 = (5/2)t - 1/2$, $x_2 = 4t - 1$, $x_3 = t$, where t is any real number

Determine the value(s) of k such that the system of linear equations has exactly one solution, no solution, infinitely many solutions equations

①
$$\begin{aligned} 4x + ky &= 6 \\ kx + y &= -3 \end{aligned}$$

Dividing the first equation by 4 produces
$$\begin{aligned} x + \tfrac{k}{4}y &= \tfrac{3}{2} \\ kx + y &= -3 \end{aligned}$$

Adding $-k$ times the first equation to the second equation produces

$$\begin{aligned} x + \tfrac{k}{4}y &= \tfrac{3}{2} \\ (1-\tfrac{k^2}{4})y &= -\tfrac{3}{2}k - 3 \end{aligned}$$

$$(1-\tfrac{k^2}{4}) = 0 \quad \Rightarrow k = \pm 2$$

$k = 2 \Rightarrow$ produces
$$\begin{aligned} x + \tfrac{1}{2}y &= \tfrac{3}{2} \\ 0 &= -6 \end{aligned}$$
$\Rightarrow$ No solution

$k=-2 \Rightarrow$ produces $\begin{array}{rcl} x + \frac{1}{2}y & = & \frac{3}{2} \\ 0 & = & 0 \end{array}$ $\Rightarrow$ Infinitely many solutions

$k \neq \pm 2 \quad \Rightarrow$ exactly one solution

② $\begin{array}{rcl} x + ky & = & 0 \\ kx + y & = & 0 \end{array}$

Adding $-k$ times the first equation to the second equation produces

$$\begin{array}{rcl} x + ky & = & 0 \\ (1-k^2)y & = & 0 \end{array}$$

$(1-k^2)=0 \quad \Rightarrow k=\pm 1$

$k=\pm 1$ produces $\begin{array}{rcl} x + y & = & 0 \\ 0 & = & 0 \end{array}$ $\Rightarrow$ Infinitely many solutions

$k \neq \pm 1 \quad \Rightarrow$ exactly one solution (trivial solution $x=y=0$)

③

$$\begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 8z &= 4 \end{aligned}$$

Reduce the system to row-echelon form

$$\begin{aligned} x + 2y + \quad kz &= 6 \\ (8-3k)z &= -14 \end{aligned}$$

$k = 8/3 \Rightarrow$ no solution

$k \neq 8/3 \Rightarrow$ infinitely many solutions

Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination

①
$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= 8 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-2)}} \begin{bmatrix} 1 & 2 & 7 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow{r_{2}^{(-1/3)}} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} x + 2y &= 7 \\ y &= 2 \end{aligned}$$

Using back-substitution you find that $x = 3$ and $y = 2$. Or using Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow x = 3 \text{ and } y = 2$$

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②
$$\begin{aligned} -3x &+ 5y = -22 \\ 3x &+ 4y = 4 \\ 4x &- 8y = 32 \end{aligned}$$

$$\begin{bmatrix} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} r_1^{(-1/3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} r_{12}^{(-3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 4 & -8 & 32 \end{bmatrix} r_{13}^{(-4)} \rightarrow$$

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix} r_2^{(1/9)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix} r_{23}^{(4/3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x - (5/3)y &= 22/3 \\ y &= -2 \end{aligned}$$

Using back-substitution you find that $x = 4$ and $y = -2$

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(3)

$$\begin{aligned} x \quad\quad\quad\quad - 3z &= -2 \\ 3x + y - 2z &= 5 \\ 2x + 2y + z &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{bmatrix} r_{12}^{(-3)} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 2 & 2 & 1 & 4 \end{bmatrix} r_{13}^{(-2)} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{bmatrix}$$

$$r_{23}^{(-2)} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{bmatrix} \quad r_{3}^{(-1/7)} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Using back-substitution you find that $x = 4$, $y = -3$ and $z = 2$

Or using Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix} r_{32}^{(-7)} \to \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} r_{31}^{(3)} \to \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow x = 4,\ y = -3 \text{ and } z = 2$$

4

$$\begin{aligned} x + y - 5z &= 3 \\ x - 2z &= 1 \\ 2x - y - z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix} r_{12}^{(-1)} \to \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 2 & -1 & -1 & 0 \end{bmatrix} r_{3}^{(-1)} \to \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$r_{13}^{(-2)} \to \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{bmatrix} r_{23}^{(3)} \to \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{rcrcrcr} x & + & y & - & 5z & = & 3 \\ & & y & - & 3z & = & 2 \\ & & & & 0 & = & 0 \end{array}$$

Choosing $z = \text{t}$ as the free variable

The solution is $x = 1 + 2t$, $y = 2 + 3t$, $z = t$, where t is any real number

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$$\begin{array}{rcrcrcr} 2x & & & + & 3z & = & 3 \\ 4x & - & 3y & + & 7z & = & 5 \\ 8x & - & 9y & + & 15z & = & 10 \end{array}$$

$$\begin{bmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} r_1^{(1/2)} \rightarrow \begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} r_{12}^{(-4)} \rightarrow \begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 0 & -3 & 1 & -1 \\ 8 & -9 & 15 & 10 \end{bmatrix}$$

$$r_{13}^{(-8)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{bmatrix} r_{23}^{(-3)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_{2}^{(-1/3)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the third row corresponds to the equation $0 = 1$, there is no solution to the original system

Graph the system of linear equations. Solve the system and interpret your answer

1. $$\begin{aligned} x + 3y &= 2 \\ -x + 2y &= 3 \end{aligned}$$

2. $$\begin{aligned} x + 3y &= 17 \\ 4x + 3y &= 7 \end{aligned}$$

3. $$\begin{aligned} \frac{x}{4} + \frac{y}{6} &= 1 \\ x - y &= 3 \end{aligned}$$

Solve the system of linear equations

1. $$\begin{aligned} 3u + v &= 240 \\ u + 3v &= 240 \end{aligned}$$

2. $$\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$$

3. $$\begin{aligned} x - y - z &= 0 \\ x + 2y - z &= 6 \\ 2x - z &= 5 \end{aligned}$$

4. $$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

5. $$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

Determine the value(s) of k such that the system of linear equations has exactly one solution, no solution, infinitely many solutions equations

① $$\begin{aligned} x + ky &= 2 \\ kx + y &= 4 \end{aligned}$$

② $$\begin{aligned} 4x + ky &= 0 \\ kx + y &= 0 \end{aligned}$$

③ $$\begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 9z &= -1 \end{aligned}$$

Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination

① $$\begin{aligned} x + 3y &= 11 \\ 3x + y &= 9 \end{aligned}$$

② $$\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$$

③ $$\begin{aligned} 2x - 2y + 3z &= 22 \\ 3y - z &= 24 \\ 6x - 7y &= -22 \end{aligned}$$

④ $$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

⑤ $$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

Find, if possible, (a) $A + B$, (b) $A - B$, (c) $2A$, (d) $2A - B$, and (e) $B + 1/2A$

① $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

$(a)\ A + B = \begin{bmatrix} -2 & 0 \\ 6 & 3 \end{bmatrix} \qquad (b)\ A - B = \begin{bmatrix} 4 & 4 \\ -2 & -1 \end{bmatrix} \qquad (c)\ 2A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

$(d)\ 2A - B = \begin{bmatrix} 5 & 6 \\ 0 & 0 \end{bmatrix} \qquad (e)\ B + \frac{1}{2}A = \begin{bmatrix} -5/2 & -1 \\ 5 & 5/2 \end{bmatrix}$

② $A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & -2 \end{bmatrix}$

$$(a)\ A+B=\begin{bmatrix}4 & -2 & 5\\ -4 & 0 & 2\end{bmatrix}\qquad (b)\ A-B=\begin{bmatrix}0 & 4 & -3\\ 2 & -2 & 6\end{bmatrix}\qquad (c)\ 2A=\begin{bmatrix}4 & 2 & 2\\ -2 & -2 & 8\end{bmatrix}$$

$$(d)\ 2A-B=\begin{bmatrix}2 & 5 & -2\\ 1 & -3 & 10\end{bmatrix}\qquad (e)\ B+\frac{1}{2}A=\begin{bmatrix}3 & -5/2 & 9/2\\ -7/2 & 1/2 & 0\end{bmatrix}$$

③ $$A=\begin{bmatrix}6 & 0 & 3\\ -1 & -4 & 0\end{bmatrix},\quad B=\begin{bmatrix}8 & -1\\ 4 & -3\end{bmatrix}$$

$$(c)\ 2A=\begin{bmatrix}12 & 0 & 6\\ -2 & -8 & 0\end{bmatrix}$$

All other operations are not defined (A and B are different sizes)

Find, if possible, (a) $\boldsymbol{AB}$, (b) $\boldsymbol{BA}$

① $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}, \quad B=\begin{bmatrix}2 & -1\\ -1 & 8\end{bmatrix}$

$$(a)\ AB=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 8\end{bmatrix}=\begin{bmatrix}1(2)+2(-1) & 1(-1)+2(8)\\ 4(2)+2(-1) & 4(-1)+2(8)\end{bmatrix}=\begin{bmatrix}0 & 15\\ 6 & 12\end{bmatrix}$$

$$(b)\ BA=\begin{bmatrix}2 & -1\\ -1 & 8\end{bmatrix}\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}=\begin{bmatrix}2(1)+(-1)(4) & 2(2)+(-1)(2)\\ -1(1)+8(4) & -1(2)+8(2)\end{bmatrix}=\begin{bmatrix}-2 & 2\\ 31 & 14\end{bmatrix}$$

② $A=\begin{bmatrix}2 & 1\\ -3 & 4\\ 1 & 6\end{bmatrix}, \quad B=\begin{bmatrix}0 & -1 & 0\\ 4 & 0 & 2\\ 8 & -1 & 7\end{bmatrix}$

(a) AB is not defined because A is 3×2 and B is 3×3

$$(b)\ BA = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0(2)+(-1)(-3)+0(1) & 0(1)+(-1)(4)+0(6) \\ 4(2)+0(-3)+2(1) & 4(1)+0(4)+2(6) \\ 8(2)+(-1)(-3)+7(1) & 8(1)+(-1)(4)+7(6) \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 10 & 16 \\ 26 & 46 \end{bmatrix}$$

3 $A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

$$(a)\ AB = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3(2)+2(3)+1(0) \end{bmatrix} = \begin{bmatrix} 12 \end{bmatrix}$$

$$(b)\ AB=\begin{bmatrix}2\\3\\0\end{bmatrix}\begin{bmatrix}3 & 2 & 1\end{bmatrix}=\begin{bmatrix}2(3) & 2(2) & 2(1)\\3(3) & 3(2) & 3(1)\\0(3) & 0(2) & 0(1)\end{bmatrix}=\begin{bmatrix}6 & 4 & 2\\9 & 6 & 3\\0 & 0 & 0\end{bmatrix}$$

4 $A=\begin{bmatrix}2 & 0\\0 & -3\end{bmatrix},\quad B=\begin{bmatrix}-5 & 0\\0 & 4\end{bmatrix}$

$$(a)\ AB=\begin{bmatrix}2 & 0\\0 & -3\end{bmatrix}\begin{bmatrix}-5 & 0\\0 & 4\end{bmatrix}=\begin{bmatrix}-10 & 0\\0 & -12\end{bmatrix}$$

$$(b)\ BA=\begin{bmatrix}-5 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\0 & -3\end{bmatrix}=\begin{bmatrix}-10 & 0\\0 & -12\end{bmatrix}$$

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$$A=\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 2\\ 0 & 0 & 3\end{bmatrix},\quad B=\begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}$$

$$(a)\ AB=\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 2\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}=\begin{bmatrix}2 & 1 & 9\\ 0 & -1 & 6\\ 0 & 0 & 12\end{bmatrix}$$

$$(b)\ BA=\begin{bmatrix}2 & -1 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 2\\ 0 & 0 & 3\end{bmatrix}=\begin{bmatrix}2 & 5 & 3\\ 0 & -1 & 8\\ 0 & 0 & 12\end{bmatrix}$$

Write the system of linear equations in the form $\boldsymbol{Ax} = \boldsymbol{b}$ and solve this matrix equation for $\boldsymbol{x}$

① $$\begin{aligned} -x + y &= 4 \\ -2x + y &= 0 \end{aligned}$$

$$\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Use Gauss-Jordan elimination on the augmented matrix

$$\begin{bmatrix} -1 & 1 & 4 \\ -2 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 8 \end{bmatrix}$$

Therefore, the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

②
$$\begin{aligned} 2x - y - z &= 0 \\ x - 2y + 2z &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Use Gauss-Jordan elimination on the augmented matrix

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ 1 & -2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & -5/3 & 0 \end{bmatrix} \Rightarrow \begin{aligned} x \quad - 4/3z &= 0 \\ y - 5/3z &= 0 \end{aligned}$$

Free variable: $x = t$

Therefore, the solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ (5/4)t \\ (3/4)t \end{bmatrix} = t\begin{bmatrix} 1 \\ 5/4 \\ 3/4 \end{bmatrix}, t \in R$

Solve for $\boldsymbol{X}$ in the equation, given

$$A=\begin{bmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}$$

(a) $3X+2A=B$ (b) $2A-5B=3X$

(c) $X-3A+2B=O$ (d) $6X-4A-3B=O$

(a) $3X+2A=B$

$$3X=B-2A=\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{bmatrix}-\begin{bmatrix} -8 & 0 \\ 2 & -10 \\ -6 & 4 \end{bmatrix}=\begin{bmatrix} 9 & 2 \\ -4 & 11 \\ 10 & 0 \end{bmatrix} \Rightarrow X=\begin{bmatrix} 3 & 2/3 \\ -4/3 & 11/3 \\ 10/3 & 0 \end{bmatrix}$$

(b) $2A - 5B = 3X$

$$3X = 2A - 5B = \begin{bmatrix} -8 & 0 \\ 2 & -10 \\ -6 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ -10 & 5 \\ 20 & 20 \end{bmatrix} = \begin{bmatrix} -13 & -10 \\ 12 & -15 \\ -26 & -16 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} -13/3 & -10/3 \\ 4 & -5 \\ -26/3 & -16/3 \end{bmatrix}$$

(c) $X - 3A + 2B = O$

$$X = 3A - 2B = \begin{bmatrix} -12 & 0 \\ 3 & -15 \\ -9 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -4 & 2 \\ 8 & 8 \end{bmatrix} = \begin{bmatrix} -14 & -4 \\ 7 & -17 \\ -17 & -2 \end{bmatrix}$$

(d) $6X - 4A - 3B = O$

$$6X = 4A + 3B = \begin{bmatrix} -16 & 0 \\ 4 & -20 \\ -12 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -6 & 3 \\ 12 & 12 \end{bmatrix} = \begin{bmatrix} -13 & 6 \\ -2 & -17 \\ 0 & 20 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} -13/6 & 1 \\ -1/3 & -17/6 \\ 0 & 10/3 \end{bmatrix}$$

Consider the matrices below

$$X=\begin{bmatrix}1\\0\\1\end{bmatrix},\quad Y=\begin{bmatrix}1\\1\\0\end{bmatrix},\quad Z=\begin{bmatrix}2\\-1\\3\end{bmatrix},\quad W=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

(a) Find scalars a and b such that $Z = aX + bY$

(b) Show that there do not exist scalars a and b such that $W = aX + bY$

(c) Show that if $aX + bY + cW = O$, then $a = 0$, $b = 0$, and $c = 0$

(d) Find scalars a, b, and c, not all equal to zero, such that $aX + bY + cZ = O$

(a) $$aX+bY=a\begin{bmatrix}1\\0\\1\end{bmatrix}+b\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}2\\-1\\3\end{bmatrix}\Rightarrow\begin{array}{rcrcr}a&+&b&=&2\\&&b&=&-1\\a&&&=&3\end{array}$$

The only solution to this system is: $a = 3$ and $b = -1$

(b) $aX + bY = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} \Rightarrow \begin{array}{ccccc} a & + & b & = & 1 \\ & & b & = & 1 \\ a & & & = & 1 \end{array}$

The system is inconsistent. No values of a and b will satisfy the equation

(c) $aX + bY + cW = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{array}{ccccccc} a & + & b & + & c & = & 0 \\ & & b & + & c & = & 0 \\ a & & & + & c & = & 0 \end{array}$

Then $a = -c$, so $b = 0$. Then $c = 0$, so $a = b = c = 0$

(d) $aX + bY + cZ = a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}2\\-1\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{array}{ccccccc} a & + & b & + & 2c & = & 0 \\ & & b & - & c & = & 0 \\ a & & & + & 3c & = & 0 \end{array}$

Using Gauss-Jordan elimination the solution is $a = -3t$, $b = t$ and $c = t$, where t is any real number. If $t = 1$, then $a = -3$, $b = 1$, and $c = 1$

Determine whether the matrix is symmetric, skew-symmetric, or neither

① $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -A$ The matrix is skew-symmetric

② $A = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix} = A$ The matrix is symmetric

Show that B is the inverse of A

① $$A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix},\quad B=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$$

$$AB=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$BA=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

② $$A=\begin{bmatrix}2 & -17 & 11\\ -1 & 11 & -7\\ 0 & 3 & -2\end{bmatrix},\quad B=\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}$$

$$AB=\begin{bmatrix}2 & -17 & 11\\ -1 & 11 & -7\\ 0 & 3 & -2\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 1 & 2\\ 2 & 4 & -3\\ 3 & 6 & -5\end{bmatrix}\begin{bmatrix}2 & -17 & 11\\ -1 & 11 & -7\\ 0 & 3 & -2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Find the inverse of the matrix (if it exists)

① $A=\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$

$$A^{-1}=\frac{1}{7-6}\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix}=\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix}$$

② $$A=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1/2 & 0\\ 0 & 1/3\end{bmatrix}$$

③ $$A=\begin{bmatrix}-1 & 1\\ 3 & -3\end{bmatrix}$$

$\det(A)=0 \Rightarrow A$ is singular and has no inverse

④ $$A=\begin{bmatrix}1 & 1 & 1\\ 3 & 5 & 4\\ 3 & 6 & 5\end{bmatrix}$$

$$[A \vdots I]=\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 3 & 5 & 4 & 0 & 1 & 0\\ 3 & 6 & 5 & 0 & 0 & 1\end{array}\right]$$

Using elementary row operations, rewrite this matrix in reduced row-echelon form

$$\left[I \,\vdots\, A^{-1}\right]=\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & -1\\ 0 & 1 & 0 & -3 & 2 & -1\\ 0 & 0 & 1 & 3 & -3 & 2\end{array}\right] \Rightarrow \quad A^{-1}=\begin{bmatrix}1 & 1 & -1\\ -3 & 2 & -1\\ 3 & -3 & 2\end{bmatrix}$$

⑤ $$A=\begin{bmatrix}1 & 2 & -1\\ 3 & 7 & -10\\ 7 & 16 & -21\end{bmatrix}$$

$$\left[A \,\vdots\, I\right]=\left[\begin{array}{ccc|ccc}1 & 2 & -1 & 1 & 0 & 0\\ 3 & 7 & -10 & 0 & 1 & 0\\ 7 & 16 & -21 & 0 & 0 & 1\end{array}\right]$$

Using elementary row operations, you cannot form the identity matrix on the left side

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 13 & 0 & -16 & 7 \\ 0 & 1 & -7 & 0 & 7 & -3 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

Therefore, the matrix is singular and has no inverse

6 $$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

Use an inverse matrix to solve each system of linear equations

1
$$\begin{aligned} x + 2y &= -1 \\ x - 2y &= 3 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{-2-2}\begin{bmatrix} -2 & -2 \\ -1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2
$$\begin{aligned} x + 2y + z &= 2 \\ x + 2y - z &= 4 \\ x - 2y + z &= -2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & -1 \\ 1 & -2 & 1 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{4}\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & -1 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The solution is: $x = 1$, $y = 1$ and $z = -1$

(a) Find $2A - A^2$, (b) Conclude A^{-1}

$$A = \begin{bmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a)

$$2A - A^2 = 2\begin{bmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 & 0 \\ 4 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} -3 & -4 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

(b)

$$2A - A^2 = A(2I - A) = I_3 \quad \Rightarrow \quad A^{-1} = (2I - A)$$

$$A^{-1} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find, if possible, (a) $\boldsymbol{A}+\boldsymbol{B}$, (b) $\boldsymbol{A}-\boldsymbol{B}$, (c) $2\boldsymbol{A}$, (d) $3\boldsymbol{A}-\boldsymbol{B}$, and (e) $2\boldsymbol{B}+1/3\boldsymbol{A}$

1. $A=\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix},\quad B=\begin{bmatrix}-1 & 2\\ -2 & -3\end{bmatrix}$

2. $A=\begin{bmatrix}2 & 1 & 3\\ -1 & -2 & 0\end{bmatrix},\quad B=\begin{bmatrix}5 & -1\\ 2 & 3\end{bmatrix}$

3. $A=\begin{bmatrix}1 & 1 & 1\\ -1 & -1 & 5\end{bmatrix},\quad B=\begin{bmatrix}2 & -2 & 4\\ -3 & 1 & 3\end{bmatrix}$

Find, if possible, (a) $\boldsymbol{AB}$, (b) $\boldsymbol{BA}$

1. $A=\begin{bmatrix}2 & -2\\ -1 & 4\end{bmatrix},\quad B=\begin{bmatrix}4 & 1\\ 2 & -2\end{bmatrix}$

2. $A=\begin{bmatrix}3 & 2 & 1\\ -3 & 0 & 4\\ 4 & -2 & -4\end{bmatrix},\quad B=\begin{bmatrix}1 & 2\\ 2 & -1\\ 1 & -2\end{bmatrix}$

3. $A=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 2\\ 0 & 0 & -2\end{bmatrix},\quad B=\begin{bmatrix}2 & 0 & 0\\ -1 & 1 & 0\\ 3 & 2 & 4\end{bmatrix}$

4. $A=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix},\quad B=\begin{bmatrix}1 & 2 & 1\end{bmatrix}$

Write the system of linear equations in the form $Ax = b$ and solve this matrix equation for

① $$\begin{aligned} 2x + 3y &= 5 \\ x + 4y &= 10 \end{aligned}$$

② $$\begin{aligned} x + y - 3z &= -1 \\ -x + 2y \qquad &= 1 \\ 2x - y + z &= 2 \end{aligned}$$

Find the inverse of the matrix (if it exists)

① $$A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

② $$A = \begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$$

③ $$A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

④ $$A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 5 & 5 \end{bmatrix}$$

Use an inverse matrix to solve each system of linear equations

①
$$\begin{aligned} 2x - y &= -3 \\ 2x + y &= 7 \end{aligned}$$

②
$$\begin{aligned} x + y - 2z &= 0 \\ x - 2y + z &= 0 \\ x - y - z &= -1 \end{aligned}$$

(a) Find $A^3 - 5A^2 + 8A$, (b) Conclude A^{-1}

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

$(A + B)^2 = A^2 + 2AB + B^2$?

$(A - B)^2 = A^2 - 2AB + B^2$?

$(A + B)(A - B) = A^2 - B^2$?

Find a sequence of elementary matrices that can be used to write the matrix in row-echelon form

① $A = \begin{bmatrix} 0 & 1 & 7 \\ 5 & 10 & -5 \end{bmatrix}$

$$A_1 = r_{12}(A) = \begin{bmatrix} 5 & 10 & -5 \\ 0 & 1 & 7 \end{bmatrix} \Rightarrow E_1 = r_{12}(I_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = r_1^{(1/5)}(A_1) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 7 \end{bmatrix} \Rightarrow E_2 = r_1^{(1/5)}(I_2) = \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{So, } B = E_2 E_1 A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 7 \end{bmatrix}$$

2

$$A=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 4 & 8 & -4\\ -6 & 12 & 8 & 1\end{bmatrix}$$

$$A_1=r_{13}^{(6)}(A)=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 4 & 8 & -4\\ 0 & 0 & 2 & 1\end{bmatrix}\Rightarrow E_1=r_{13}^{(6)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 6 & 0 & 1\end{bmatrix}$$

$$A_2=r_{2}^{(1/4)}(A_1)=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 1 & 2 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}\Rightarrow E_2=r_{2}^{(1/4)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A_3=r_{3}^{(1/2)}(A_2)=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 1 & 2 & -1\\ 0 & 0 & 1 & 1/2\end{bmatrix}\Rightarrow E_3=r_{3}^{(1/2)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}$$

So, $B = E_3E_2E_1A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$

Find the inverse of the matrix using elementary matrices

① $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form

$$A_1 = r_{12}(A) = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \Rightarrow E_1 = r_{12}(I_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = r_{12}^{(-3)}(A_1) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow E_2 = r_{12}^{(-3)}(I_2) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A_3 = r_2^{(-1/2)}(A_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E_3 = r_{12}^{(-1/2)}(I_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

So, $B = I_2 = E_3E_2E_1A \Rightarrow A^{-1} = E_3E_2E_1$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/2 & 3/2 \end{bmatrix}$$

② $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 6 & -1 \\ 0 & 0 & 4 \end{bmatrix}$

$$A_1 = r_2^{(1/6)}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow E_1 = r_2^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_3^{(1/4)}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = r_3^{(1/4)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$A_3 = r_{31}^{(1)}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = r_{31}^{(1)}(I_3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = r_{32}^{(1/6)}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_4 = r_{32}^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $B = I_3 = E_4E_3E_2E_1A \Rightarrow A^{-1} = E_4E_3E_2E_1$

$$
\begin{aligned}
A^{-1} &= E_4E_3E_2E_1 \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/24 \\ 0 & 0 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1/6 & 1/24 \\ 0 & 0 & 1/4 \end{bmatrix}
\end{aligned}
$$

Find a sequence of elementary matrices whose product is the given nonsingular matrix

① $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form

$$A_1 = r_{12}(A) = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \Rightarrow E_1 = r_{12}(I_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = r_{12}^{(-3)}(A_1) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow E_2 = r_{12}^{(-3)}(I_2) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A_3 = r_2^{(-1/2)}(A_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E_3 = r_{12}^{(-1/2)}(I_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$\text{So, } B = I_2 = E_3E_2E_1A \quad \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

② $$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 6 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A_1 = r_2^{(1/6)}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow E_1 = r_2^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_3^{(1/4)}(A_1) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = r_3^{(1/4)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$A_3 = r_{31}^{(1)}(A_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = r_{31}^{(1)}(I_3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = r_{32}^{(1/6)}(A_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_4 = r_{32}^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $B = I_3 = E_4 E_3 E_2 E_1 A \quad \Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix}$$

Find an LU-factorization of the matrix

① $A = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$

$$A_1 = r_{12}^{(-2)}(A) = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix} \Rightarrow E_1 = r_{12}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_{13}^{(1)}(A_1) = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow E_2 = r_{13}^{(1)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A_3 = r_{23}^{(-1)}(A_2) = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow E_3 = r_{23}^{(-1)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

So, $U = E_3E_2E_1A \quad \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}U = LU$

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = LU$$

Use an LU-factorization of the coefficient matrix to solve the linear system

①
$$\begin{aligned} 2x + y \quad &= 1 \\ y - z &= 2 \\ -2x + y + z &= -2 \end{aligned}$$

$$A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$A_1 = r_{13}^{(1)}(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \Rightarrow E_1 = r_{13}^{(1)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_{23}^{(-2)}(A_1) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow E_2 = r_{23}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

So, $U = E_2 E_1 A \quad \Rightarrow A = E_1^{-1} E_2^{-1} U = LU$

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$A=\begin{bmatrix}1&0&0\\0&1&0\\-1&2&1\end{bmatrix}\begin{bmatrix}2&0&1\\0&1&-1\\0&0&3\end{bmatrix}=LU$$

$L\boldsymbol{y}=\boldsymbol{b}:$

$$\begin{bmatrix}1&0&0\\0&1&0\\-1&2&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\2\\-2\end{bmatrix}\quad\Rightarrow y_1=1,\, y_2=2,\, y_3=-5$$

$U\boldsymbol{x}=\boldsymbol{y}:$

$$\begin{bmatrix}2&0&1\\0&1&-1\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\-5\end{bmatrix}\quad\Rightarrow x_1=\frac{1}{3},\, x_2=\frac{1}{3},\, x_3=-\frac{5}{3}$$

Find a sequence of elementary matrices that can be used to write the matrix in row-echelon form

① $A=\begin{bmatrix}1 & 3 & 0\\ 2 & 5 & -1\\ 3 & -2 & -4\end{bmatrix}$

② $A=\begin{bmatrix}0 & 3 & -3 & 6\\ 1 & -1 & 2 & -2\\ 0 & 0 & 2 & 2\end{bmatrix}$

Find the inverse of the matrix using elementary matrices

① $A=\begin{bmatrix}3 & -1\\ 1 & -1\end{bmatrix}$

② $A=\begin{bmatrix}2 & 3 & 1\\ 2 & -3 & -3\\ 4 & 0 & 3\end{bmatrix}$

Find a sequence of elementary matrices whose product is the given nonsingular matrix

① $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

Find an LU-factorization of the matrix

① $A = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$

Use an LU-factorization of the coefficient matrix to solve the linear system

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$$\begin{aligned} 2x + y \quad &= 1 \\ y - z &= 2 \\ -2x + y + z &= -2 \end{aligned}$$