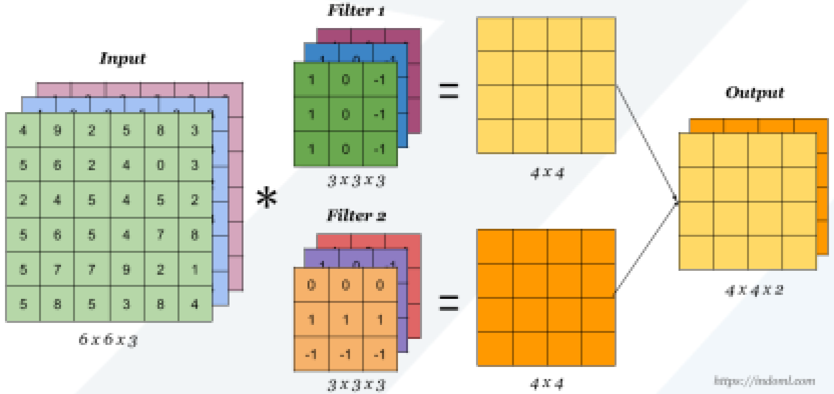
CECC102, CECC122 & CEDC102 : Linear Algebra (and Matrix Theory)

Exercises 3: Determinants



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Use expansion by cofactors to find the determinant of the matrix

① $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$ ② $\begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 1 & 1 & -5 \end{bmatrix}$ ③ $\begin{bmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = -3\begin{vmatrix} 4 & -2 \\ 4 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & 4 \\ -1 & 4 \end{vmatrix} = -3(20) + 2(1) = -58$$

$$\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 1 & 1 & -5 \end{vmatrix} = 2\begin{vmatrix} 3 & 1 \\ 1 & -5 \end{vmatrix} - 0\begin{vmatrix} 4 & 6 \\ 1 & -5 \end{vmatrix} + 1\begin{vmatrix} 4 & 6 \\ 3 & 1 \end{vmatrix} = 2(-15-1) + 1(4-18) = -46$$

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} x & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix} = x - 2 + 3x - 2y = 4x - 2y - 2$$

Use elementary row or column operations to find the determinant

① $\begin{bmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{bmatrix}$ ② $\begin{bmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}$ ③ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{bmatrix}$ ④ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 3 & -1 \end{bmatrix}$

$r_{12}^{(-1)}, r_{13}^{(-4)}, r_{23}^{(-5)}$

$$\begin{vmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -3 \\ 0 & -4 & 4 \\ 4 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -3 \\ 0 & -4 & 4 \\ 0 & -20 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & -7 \end{vmatrix} = (1)(-4)(-7) = 28$$

$$r_{12},\ r_{12}^{(-2)},\ r_{13}^{(-1)}$$

$$\begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -5 \\ 0 & -2 & 1 \end{vmatrix} = -(1)\begin{vmatrix} -7 & -5 \\ -2 & 1 \end{vmatrix} = -(1)(-7-10) = 17$$

$$c_{31}^{(1)}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -2 & -1 \end{vmatrix} = (2)^{1+1}\begin{vmatrix} -1 & -2 \\ -2 & -1 \end{vmatrix} = (2)(1-4) = -6$$

$$r_{23}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 3 & -1 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{vmatrix} = -(1)(3)(4) = -12$$

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Solve the system of linear equations by each of the methods listed below.

(a) Gaussian elimination with back-substitution

(b) Gauss-Jordan elimination

(c) Cramer’s Rule

1

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -7 \\ 2x_1 - 2x_2 - 2x_3 &= -8 \\ -x_1 + 3x_2 + 4x_3 &= 8 \end{aligned}$$

(a)

$$\begin{bmatrix} 1 & 2 & -1 & -7 \\ 2 & -2 & -2 & -8 \\ -1 & 3 & 4 & 8 \end{bmatrix} r_{12}^{(-2)}, r_{13}^{(1)} \begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & -6 & 0 & 6 \\ 0 & 5 & 3 & 1 \end{bmatrix} r_2^{(-1/6)} \begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 5 & 3 & 1 \end{bmatrix} r_{23}^{(-5)}$$

$$\begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 6 \end{bmatrix} r_2^{(\frac{1}{3})} \begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So, $x_3 = 2$, $x_2 = -1$, and $x_1 = -7 + 1(2) - 2(-1) = -3$

(b) $\begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} r_{21}^{(-2)} \begin{bmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} r_{31}^{(1)} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

So, $x_1 = -3$, $x_2 = -1$, and $x_3 = -3$

(c)

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & 3 & 4 \end{bmatrix} \Rightarrow |A| = -18 \qquad A_1 = \begin{bmatrix} -7 & 2 & -1 \\ -8 & -2 & -2 \\ 8 & 3 & 4 \end{bmatrix} \Rightarrow |A_1| = 54$$

$$A_2 = \begin{bmatrix} 1 & -7 & -1 \\ 2 & -8 & -2 \\ -1 & 8 & 4 \end{bmatrix} \Rightarrow |A_2| = 18 \qquad A_3 = \begin{bmatrix} 1 & 2 & -7 \\ 2 & -2 & -8 \\ -1 & 3 & 8 \end{bmatrix} \Rightarrow |A_3| = -36$$

So, $x_1 = \dfrac{54}{-18} = -3, \quad x_2 = \dfrac{18}{-18} = -1, \quad x_3 = \dfrac{-36}{-18} = 2$

Find the value(s) of k such that A is singular

① $\begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix}$

$$|A| = 0 \Rightarrow \begin{vmatrix} k-1 & 3 \\ 2 & k-2 \end{vmatrix} = (k-1)(k-2) - 2(3) = k^2 - 3k - 4 = 0 \Rightarrow k = -1, 4$$

② $\begin{bmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{bmatrix}$

$$|A| = 0 \Rightarrow \begin{vmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{vmatrix} = -k\begin{vmatrix} -2 & -k \\ 3 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ -2 & -k \end{vmatrix} = -k(8+3k) - (-k+4) = 0$$

$$-3k^2 - 7k - 4 = 0 \Rightarrow k = -4/3, -1$$

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Find (a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$

1 $$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$|A| = 5(-3)(2) = -30$$

(a) $|A^T| = |A| = -30$

(b) $|A^2| = |A|^2 = (-30)^2 = 900$

(c) $|AA^T| = |A||A^T| = (-30)(-30) = 900$

(d) $|2A| = 2^3|A| = 8(-30) = -240$

(e) $|A^{-1}| = \dfrac{1}{|A|} = \dfrac{-1}{30}$

Use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution

①
$$\begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}$$

②
$$\begin{aligned} x + 2y + 6z &= 1 \\ 2x + 5y + 15z &= 4 \\ 3x + y + 3z &= -6 \end{aligned}$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 2 & 3 & 1 \\ 5 & 4 & 2 \end{vmatrix} = -15 \neq 0$$ The system has a unique solution

$$\begin{vmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 3 & 1 & 3 \end{vmatrix} = 0$$ The system does not have a unique solution

Use expansion by cofactors to find the determinant of the matrix

① $\begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}$ ② $\begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$ ③ $\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Use elementary row or column operations to find the determinant

① $\begin{bmatrix} 1 & -3 & 2 \\ 5 & 2 & -1 \\ -1 & 0 & 6 \end{bmatrix}$ ② $\begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & 6 \end{bmatrix}$ ③ $\begin{bmatrix} 3 & -1 & -3 \\ -1 & -4 & -2 \\ 3 & -1 & -1 \end{bmatrix}$

Solve the system of linear equations by each of the methods listed below.

(a) Gaussian elimination with back-substitution

(b) Gauss-Jordan elimination

(c) Cramer's Rule

1
$$\begin{aligned} 2x_1 + x_2 + 2x_3 &= 6 \\ -x_1 + 2x_2 - 3x_3 &= 0 \\ 3x_1 + 2x_2 - x_3 &= 6 \end{aligned}$$

2
$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 4 \\ 3x_1 + 5x_2 + 9x_3 &= 7 \\ 5x_1 + 9x_2 + 13x_3 &= 17 \end{aligned}$$

Find the value(s) of k such that A is singular

1
$$\begin{bmatrix} 0 & k & 1 \\ k & 1 & k \\ 1 & k & 0 \end{bmatrix}$$

2
$$\begin{bmatrix} k & -3 & -k \\ -2 & k & 1 \\ k & 1 & 0 \end{bmatrix}$$

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Find (a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$

① $\begin{bmatrix} -2 & 6 \\ 1 & 3 \end{bmatrix}$

② $\begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

Find the adjoint of the matrix A. Then use the adjoint to find the inverse of A (if possible)

① $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

② $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$