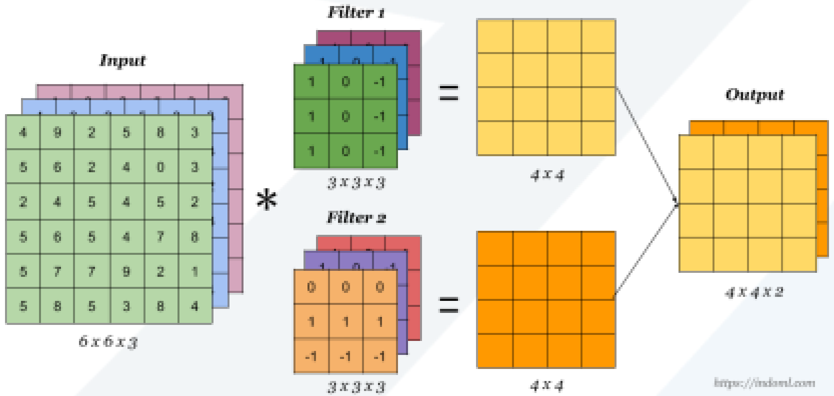
CECC102, CECC122 & CEDC102 : Linear Algebra (and Matrix Theory)

Exercises 4: Euclidean Vector Spaces



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Determine whether W is a subspace of the vector space V

① $W = \{(x, y): x = 2y\}, \; V = R^2$

W is nonempty and $W \subset R^2$, W is closed under addition and scalar multiplication
$\Rightarrow$ W is a subspace of R^2

② $W = \{(x, 2x, 3x): x \text{ is a real number}\}, \; V = R^3$

W is nonempty and $W \subset R^3$, W is closed under addition and scalar multiplication
$\Rightarrow$ W is a subspace of R^3

③ $W = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 0\}$ $\qquad$ $W = \{\mathbf{0}\}$, W is a subspace of R^3

④ $W = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 1\}$

W is not closed under addition or scalar multiplication, so it is not a subspace of R^3.
$(1, 0, 0) \in W$, and yet $2(1, 0, 0) = (2, 0, 0) \notin W$

Write each vector as a linear combination of the vectors in S (if possible)

① $S = \{(2, -1, 3), (5, 0, 4)\}$

(a) $\boldsymbol{u} = (1, 1, -1)$

(b) $\boldsymbol{v} = (8, -1/4, 27/4)$

(a) $\boldsymbol{u} = (1, 1, -1) = c_1(2, -1, 3) + c_2(5, 0, 4)$

$$\begin{aligned} 2c_1 + 5c_2 &= 1 \\ -c_1 \qquad &= 1 \\ 3c_1 + 4c_2 &= -1 \end{aligned}$$

This system has no solution. So, $\boldsymbol{u}$ cannot be written as a linear combination of vectors in S

(b) $\boldsymbol{v} = (8, -1/4, 27/4) = c_1(2, -1, 3) + c_2(5, 0, 4)$

$$2c_1 + 5c_2 = 8$$
$$-c_1 = -1/4$$
$$3c_1 + 4c_2 = 27/4$$

The solution to this system is $c_1 = 1/4$ and $c_2 = 3/2$. So, $\boldsymbol{v}$ can be written as a linear combination of vectors in S

② $S = \{(2, 0, 7), (2, 4, 5), (2, -12, 13)\}$

(a) $\boldsymbol{u} = (-1, 5, -6)$

(b) $\boldsymbol{v} = (-3, 15, 18)$

(a) $\boldsymbol{u} = (-1, 5, -6) = c_1(2, 0, 7) + c_2 (2, 4, 5) + c_3 (2, -12, 13)$

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= -1 \\ 4c_2 - 12c_3 &= 5 \\ 7c_1 + 5c_2 + 13c_3 &= -6 \end{aligned}$$

One solution to this system is $c_1 = -7/4$, $c_2 = 5/4$ and $c_3 = 0$. So, $\boldsymbol{u}$ can be written as a linear combination of vectors in S

(b) $\boldsymbol{v} = (-3, 15, 18) = c_1(2, 0, 7) + c_2\,(2, 4, 5) + c_3\,(2, -12, 13)$

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= -3 \\ 4c_2 - 12c_3 &= 15 \\ 7c_1 + 5c_2 + 13c_3 &= 18 \end{aligned}$$

This system has no solution. So, $\boldsymbol{v}$ cannot be written as a linear combination of vectors in S

Determine whether the set S is linearly independent or linearly dependent

① $S = \{(-2, 2), (3, 5)\}$

$(-2, 2)$ is not a scalar multiple of $(3, 5)$. So the set S is linearly independent

② $S = \{(0, 0), (1, -1)\}$

$\mathbf{0} \in S \Rightarrow S$ is linearly dependent

③ $S = \{(3, -6), (-1, 2)\}$

$(3, -6) = -3\,(-1, 2) \Rightarrow S$ is linearly dependent

④ $S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$

These vectors are multiples of each other, the set S is linearly dependent

⑤ $S = \{(-2, 1, 3), (2, 9, -3), (2, 3, -3)\}$

$c_1(-2, 1, 3) + c_2\,(2, 9, -3) + c_3(2, 3, -3) = \mathbf{0} = (0, 0, 0)$

$$\begin{aligned} -2c_1 + 2c_2 + 2c_3 &= 0 \\ c_1 + 9c_2 + 3c_3 &= 0 \\ 3c_1 - 3c_2 - 3c_3 &= 0 \end{aligned}$$

The system has many solutions. One solution is (3, –2, 5), so S is linearly dependent

⑥ $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$

$c_1(-4, -3, 4) + c_2\,(1, -2, 3) + c_3(6, 0, 0) = \mathbf{0} = (0, 0, 0)$

$$\begin{aligned} -4c_1 + c_2 + 6c_3 &= 0 \\ -3c_1 - 2c_2 &= 0 \\ 4c_1 + 3c_2 &= 0 \end{aligned}$$

This system has only the trivial solution $c_1 = c_2 = c_3 = 0$. So S is linearly independent

7 $S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$

$(1, 5, -3) = (1, 0, 0) + 5/4(0, 4, 0) + 1/2(0, 0, -6)$

The fourth vector is a linear combination of the first three. So S is linearly dependent

Let v_1, v_2, and v_3 be three linearly independent vectors in a vector space V. Is the set $\{v_1 - 2v_2, 2v_2 - 3v_3, 3v_3 - v_1\}$ linearly dependent or linearly independent? Explain

To see if the given set is linearly independent, solve the equation

$$c_1(\boldsymbol{v}_1 - 2\boldsymbol{v}_2) + c_2(2\boldsymbol{v}_2 - 3\boldsymbol{v}_3) + c_3(3\boldsymbol{v}_3 - \boldsymbol{v}_1) = \mathbf{0}$$

$$(c_1 - c_3)\boldsymbol{v}_1 + (-2c_1 + 2c_2)\boldsymbol{v}_2 + (-3c_2 + 3c_3)\boldsymbol{v}_3 = \mathbf{0}$$

$\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be three linearly independent vectors $\Rightarrow$

$$\begin{aligned} c_1 \qquad\quad - c_3 &= 0 \\ -2c_1 + 2c_2 \qquad\quad &= 0 \\ -3c_2 + 3c_3 &= 0 \end{aligned}$$

This system has infinitely many solutions, so $\{\boldsymbol{v}_1 - 2\boldsymbol{v}_2, 2\boldsymbol{v}_2 - 3\boldsymbol{v}_3, 3\boldsymbol{v}_3 - \boldsymbol{v}_1\}$ is linearly dependent

Let v_1, v_2, and v_3 be three linearly independent vectors in a vector space V. Is the set $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ linearly dependent or linearly independent? Explain

To see if the given set is linearly independent, solve the equation

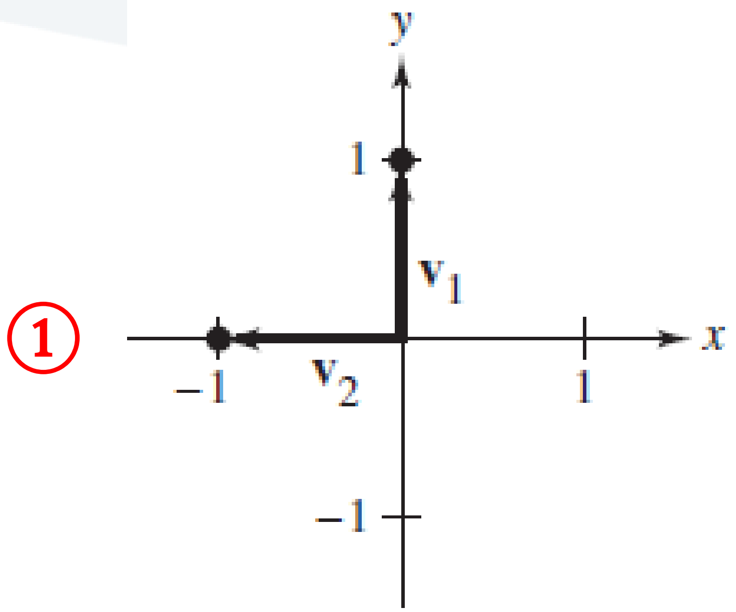
$$c_1(\mathbf{v}_1 + \mathbf{v}_2) + c_2(\mathbf{v}_2 + \mathbf{v}_3) + c_3(\mathbf{v}_3 + \mathbf{v}_1) = \mathbf{0}$$
$$(c_1 + c_3)\mathbf{v}_1 + (c_1 + c_2)\mathbf{v}_2 + (c_2 + c_3)\mathbf{v}_3 = \mathbf{0}$$

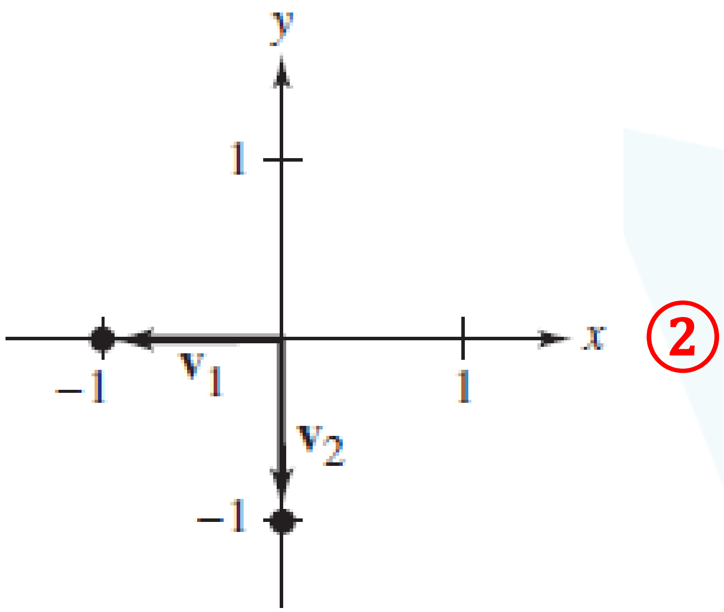
$\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be three linearly independent vectors $\Rightarrow$

$$\begin{aligned} c_1 \qquad + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

This system has only the trivial solution, so
$\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$ is linearly independent

Determine whether the set $\{v_1, v_2\}$ is a basis for R^2





1. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2
2. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2

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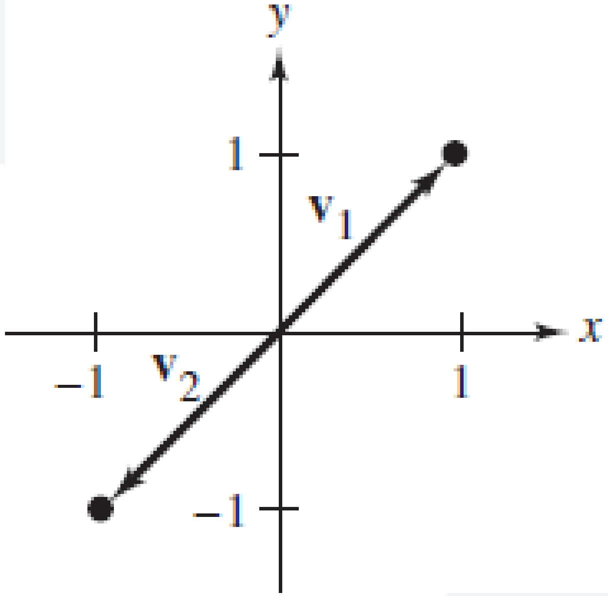

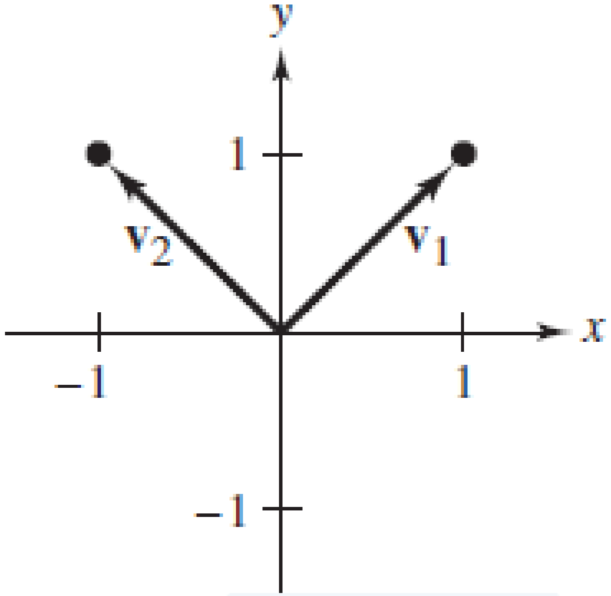

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3. v_1 and v_2 are multiplies of each other, they do not form a basis for R^2
4. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2

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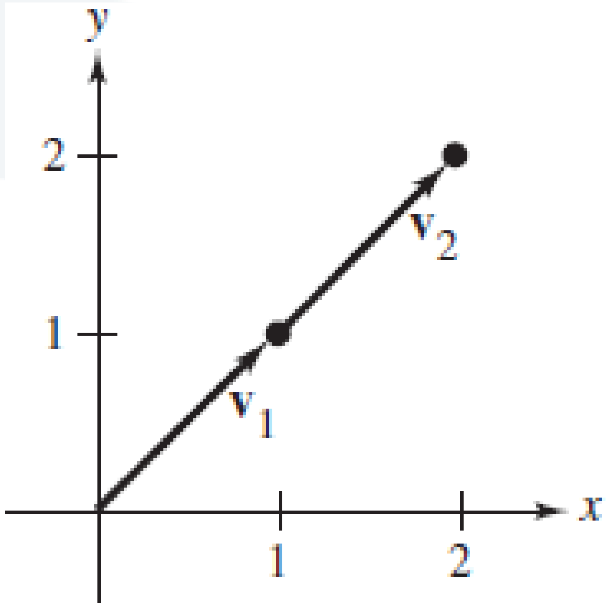

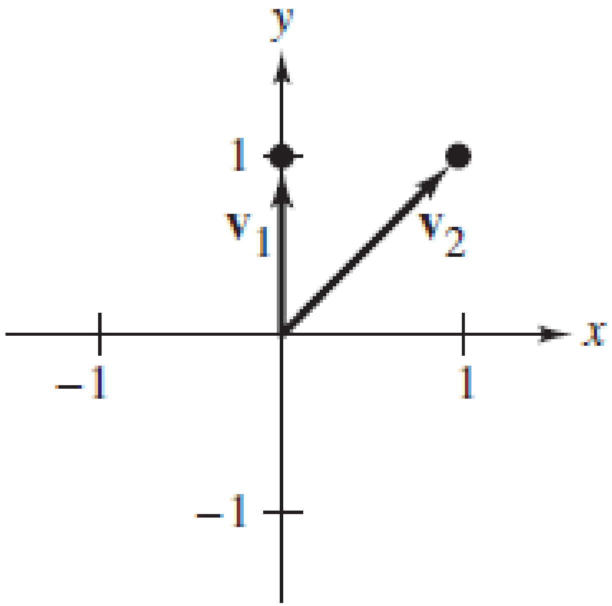

6

5. v_1 and v_2 are multiplies of each other, they do not form a basis for R^2

6. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2

Determine whether S is a basis for the given vector space

① $S = \{(4, -3), (5, 2)\}$ for R^2

S consists of exactly two linearly independent vectors, it is a basis for R^2

② $S = \{(1, 2), (1, -1), (-1, 2)\}$ for R^2

S consists of more than two vectors, so S is linearly dependent, and it is not a basis for R^2

③ $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ for R^3

To determine if the vectors in S are linearly independent, find the solution to

$$c_1(1, 5, 3) + c_2\,(0, 1, 2) + c_3(0, 0, 6) = (0, 0, 0)$$

Which corresponds to the solution of

$$\begin{aligned} c_1 &= 0 \\ 5c_1 + c_2 &= 0 \\ 3c_1 + 2c_2 + 6c_3 &= 0 \end{aligned}$$

This system has only the trivial solution. So, S consists of exactly three linearly independent vectors, and is, therefore, a basis for R^3

④ $S = \{(2, 1, 0), (0, -1, 1)\}$ for R^3

S does not span R^3 (consists of less than three vectors), although it is linearly independent $\Rightarrow$ S is not a basis for R^3

⑤ $S = \{(0, 3, -2), (4, 0, 3), (-8, 15, -16)\}$ for R^3

To determine if the vectors in S are linearly independent, find the solution to

$c_1(0, 3, -2) + c_2(4, 0, 3) + c_3(-8, 15, -16) = (0, 0, 0)$

which corresponds to the solution of

$$\begin{aligned} 4c_2 - 8c_3 &= 0 \\ 3c_1 + 15c_3 &= 0 \\ -2c_1 + 3c_2 - 16c_3 &= 0 \end{aligned}$$

This system has nontrivial solutions (for instance, $c_1 = -5$, $c_2 = 2$ and $c_3 = 1$), so the vectors are linearly dependent, and S is not a basis for R^3

6. $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ for R^3

This set contains the zero vector, and is, therefore, linearly dependent. So, S is not a basis for R^3

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Determine whether the set, (a) is linearly independent, and (b) is a basis for $\boldsymbol{R}^3$

① $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$

(a) $c_1(1, -5, 4) + c_2(11, 6, -1) + c_3(2, 3, 5) = (0, 0, 0)$

$$\begin{aligned} c_1 + 11c_2 + 2c_3 &= 0 \\ -5c_1 + 6c_2 + 3c_3 &= 0 \\ 4c_1 - c_2 + 5c_3 &= 0 \end{aligned}$$

This system has only the trivial solution. So, S is linearly independent.

(b) S consists of exactly three linearly independent vectors, and is, therefore, a basis for R^3

② $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$

(a) S is linearly dependent because the 4th vector is a linear combination of the first three $(-1, 2, -3) = -1(1, 0, 0) + 2(0, 1, 0) - 3(0, 0, 1)$

(b) S is not a basis because it is not linearly independent

Determine whether W is a subspace of the vector space V

1. $W = \{(x, y): x - y = 1\}, \; V = R^2$

2. Which of the subsets of R^3 is a subspace of R^3?

(a) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 0\}$

(b) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 1\}$

Write v as a linear combination of u_1, u_2, and u_3, if possible

1. $\boldsymbol{v} = (3, 0, -6),\ \boldsymbol{u}_1 = (1, -1, 2),\ \boldsymbol{u}_2 = (2, 4, -2),\ \boldsymbol{u}_3 = (1, 2, -4)$

2. $\boldsymbol{v} = (4, 4, 5),\ \boldsymbol{u}_1 = (1, 2, 3),\ \boldsymbol{u}_2 = (-2, 0, 1),\ \boldsymbol{u}_3 = (1, 0, 0)$

Determine whether the set S is linearly independent or linearly dependent

1. $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$

2. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$

3. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$

4. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$

5. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$

Determine whether the set, (a) is linearly independent, and (b) is a basis for $\boldsymbol{R}^3$

1. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$

2. $S = \{(-1/2, 3/4, -1), (5, 2, 3), (-4, 6, -8)\}$

3. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$

4. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$