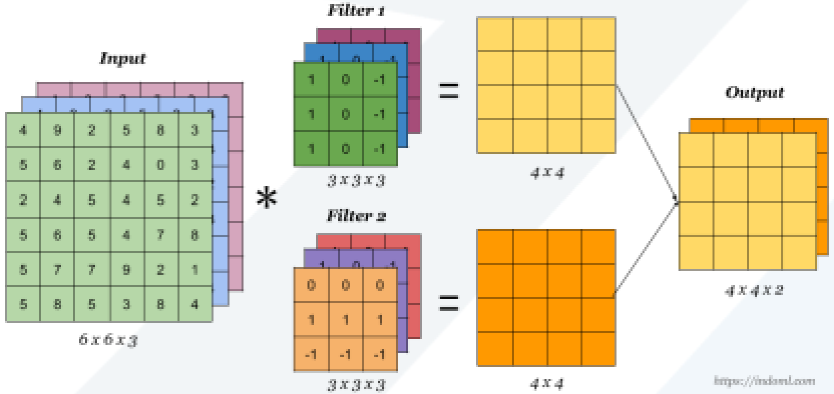
CECC102, CECC122 & CEDC102 : Linear Algebra (and Matrix Theory)

Exercises 5: General Vector Spaces



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Find the **rank** and nullity of the matrix A

① $A = \begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2,$
$\text{nullity}(A) = 4 - 2 = 2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2,$
$\text{nullity}(A) = 3 - 2 = 1$

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Given the coordinate matrix of x relative to a (nonstandard) basis B for R^n, find the coordinate matrix of x relative to the standard basis

① $B = \{(1, 1), (-1, 1)\}$, $[\boldsymbol{x}]_B = [3 \;\; 5]^T$

$\boldsymbol{x} = 3(1, 1) + 5(-1, 1) = (-2, 8)$

$(-2, 8) = -2(1, 0) + 8(0, 1)$, the coordinate vector of $\boldsymbol{x}$ relative to the standard basis is $[\boldsymbol{x}]_S = [-2 \;\; 8]^T$

② $B = \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\}$, $[\boldsymbol{x}]_B = [2 \;\; 0 \;\; -1]^T$

$\boldsymbol{x} = 2(1, 0, 0) + 0(1, 1, 0) - 1(0, 1, 1) = (2, -1, -1)$

$(-2, -1, -1) = 2(1, 0, 0) - 1(0, 1, 0) - 1(0, 0, 1)$, the coordinate vector of $\boldsymbol{x}$ relative to the standard basis is $[\boldsymbol{x}]_B = [2 \;\; -1 \;\; -1]^T$

Find the coordinate matrix of $\boldsymbol{x}$ in $\boldsymbol{R}^n$ relative to the basis B'

① $B' = \{(5, 0), (0, -8)\}, \boldsymbol{x} = (2, 2)$

$$c_1(5, 0) + c_2(0, -8) = (2, 2)$$

The resulting system of linear equations is

$$\begin{aligned} 5c_1 \quad\quad &= 2 \\ -8c_2 &= 2 \end{aligned}$$

So $c_1 = 2/5,\ c_2 = -1/4 \Rightarrow [\boldsymbol{x}]_{B'} = [2/5 \ \ -1/4]^T$

② $B' = \{(1, 2, 3), (1, 2, 0), (0, -6, 2)\}$, $\boldsymbol{x} = (3, -3, 0)$

$$c_1(1, 2, 3) + c_2(1, 2, 0) + c_3(0, -6, 2) = (3, -3, 0)$$

$$\begin{aligned} c_1 + c_2 \qquad\qquad &= 3 \\ 2c_1 + 2c_2 - 6c_3 &= -3 \\ 3c_1 \qquad\quad + 2c_3 &= 0 \end{aligned}$$

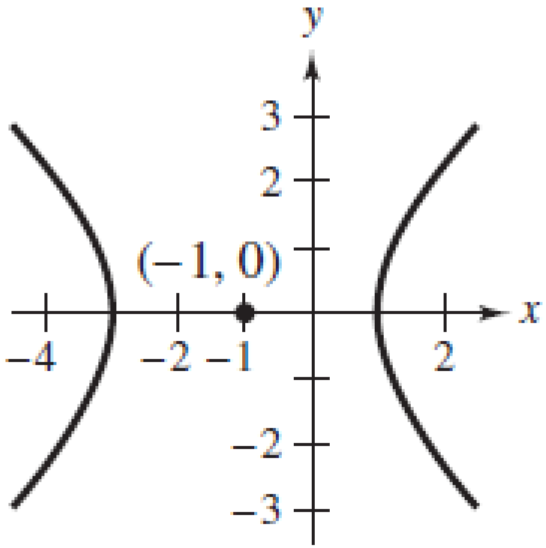
The solution is $c_1 = -1$, $c_2 = 4$ and $c_3 = 3/2 \Rightarrow [\boldsymbol{x}]_{B'} = [-1 \;\; 4 \;\; 3/2]^T$

Identify and sketch the graph of the conic section

① $x^2 - y^2 + 2x - 3 = 0$

$$(x+1)^2 - 1 - y^2 - 3 = 0$$

$$(x+1)^2 - y^2 = 4 \quad \Rightarrow \quad \frac{(x+1)^2}{2^2} - \frac{y^2}{2^2} = 1$$



(2) $16x^2 + 25y^2 - 32x - 50y + 16 = 0$

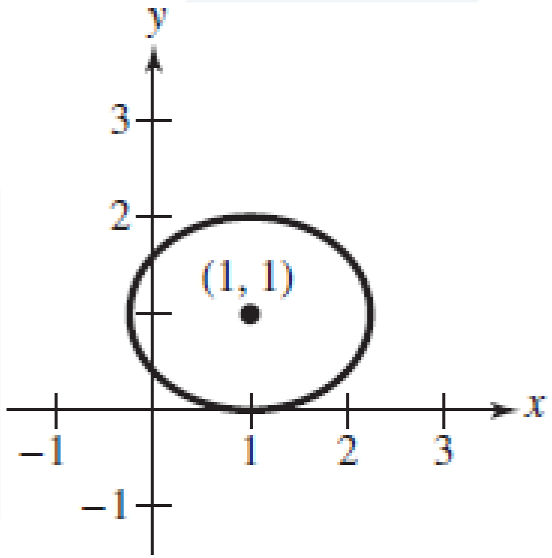
$$16(x^2 - 2x) + 25(y^2 - 2y) + 16 = 0$$

$$16(x-1)^2 - 16 + 25(y-1)^2 - 25 + 16 = 0$$

$$16(x-1)^2 + 25(y-1)^2 = 25$$

$$\frac{(x-1)^2}{25/16} - (y-1)^2 = 1$$

This is the equation of an ellipse centered at (1, 1) and a = 5/4, b = 1



Perform a rotation of axes to eliminate the xy-term, and sketch the graph of the conic

$$7x^2 + 6\sqrt{3}xy + 13y^2 - 16 = 0$$

$$\cot 2\theta = \frac{a-c}{b} = \frac{7-13}{6\sqrt{3}} = \frac{-1}{\sqrt{3}} \Rightarrow \theta = -\frac{\pi}{6}$$

$$\Rightarrow \sin\theta = -\frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}$$

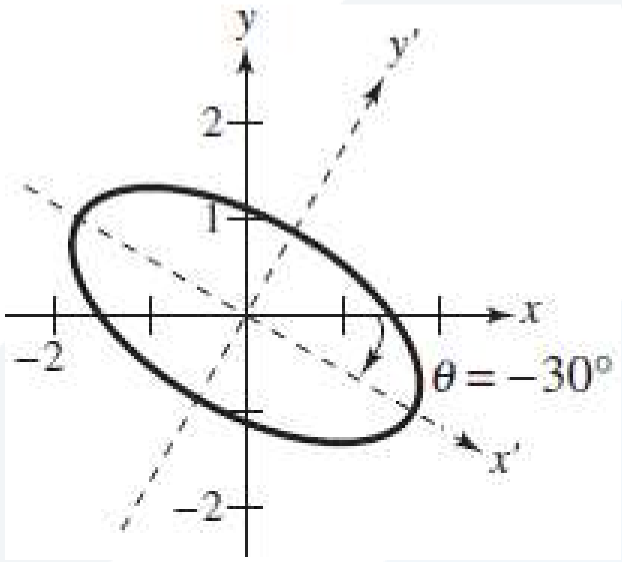
$$x = x'\cos\theta - y'\sin\theta = \frac{1}{2}(\sqrt{3}x' + y')$$

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{2}(\sqrt{3}y' - x')$$

$$7x^2 + 6\sqrt{3}xy + 13y^2 - 16 = 0 \Rightarrow 4(x')^2 + 16(y')^2 = 16$$

$$\frac{(x')^2}{4} + (y')^2 = 1$$

Ellipse centered at (0, 0) with the major axis along the x'-axis and $a = 2,\ b = 1$



Given the coordinate matrix of x relative to a (nonstandard) basis B for R^n, find the coordinate matrix of x relative to the standard basis

1. $B = \{(2, 4), (-1, 1)\}$, $[\boldsymbol{x}]_B = [4 \;\; -7]^T$

2. $B = \{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}$, $[\boldsymbol{x}]_B = [4 \;\; 0 \;\; 2]^T$

Find the **rank** and nullity of the matrix A

① $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 4 & -2 & 4 & -2 \\ -2 & 0 & 1 & 3 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 4 & 0 & 3 \\ -2 & 3 & 0 & 2 \\ 1 & 2 & 6 & 1 \end{bmatrix}$

Find the coordinate matrix of $\boldsymbol{x}$ in $\boldsymbol{R}^n$ relative to the basis B'

1. $B' = \{(2, 2), (0, -1)\}$, $\boldsymbol{x} = (-1, 2)$

2. $B' = \{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}$, $\boldsymbol{x} = (4, -2, 9)$

Identify and sketch the graph of the conic section

1. $x^2 + y^2 + 4x - 2y - 11 = 0$

2. $9x^2 + 9y^2 + 18x - 18y + 14 = 0$

3. $2x^2 - 20x - y + 46 = 0$

4. $4x^2 + y^2 + 32x + 4y + 63 = 0$

Perform a rotation of axes to eliminate the xy-term, and sketch the graph of the conic

1. $xy = 3$

2. $9x^2 + 4xy + 9y^2 - 20 = 0$

3. $x^2 + 2xy + y^2 + \sqrt{2}x - \sqrt{2}y = 0$