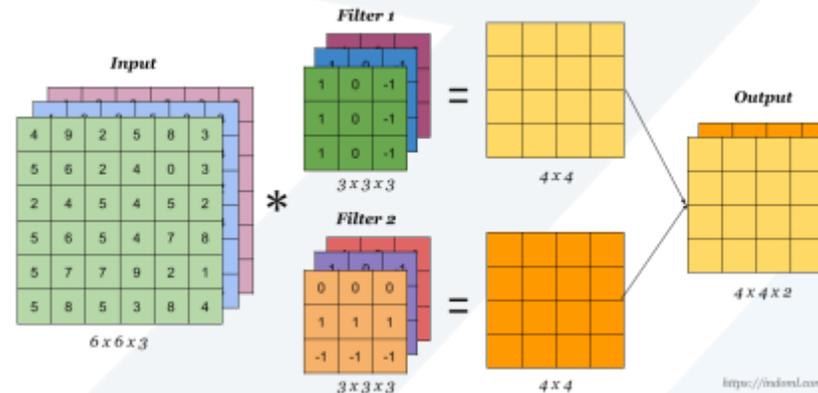


CECC102, CECC122 & CEDC102 : Linear Algebra (and Matrix Theory)

Lecture Notes 8: Eigenvalues and Eigenvectors



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Chapter 7

Eigenvalues and Eigenvectors

- 1 Eigenvalues and Eigenvectors
- 2 Diagonalization
- 3 Symmetric Matrices and Orthogonal Diagonalization
- 4 Singular Value Decomposition



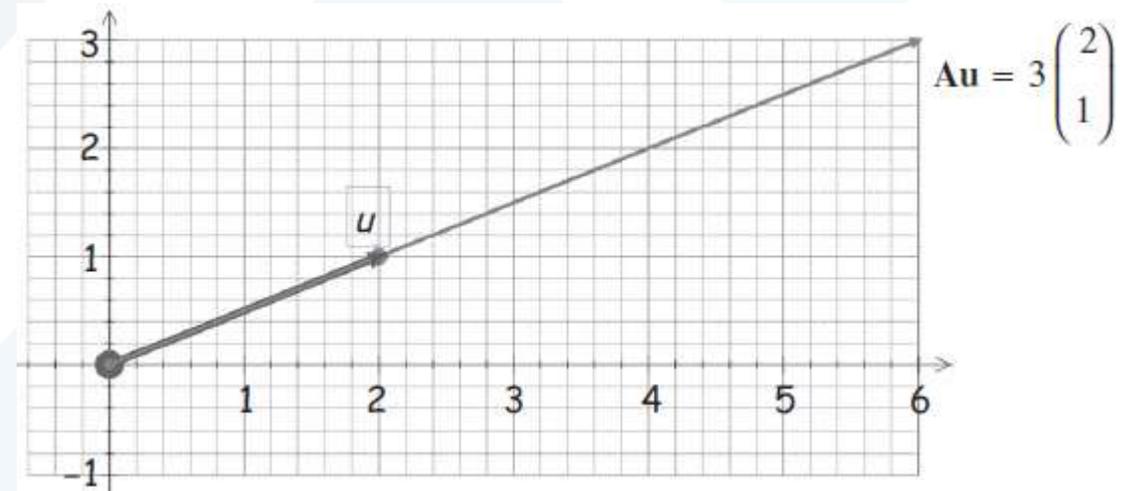
1. Eigenvalues and Eigenvectors

- By knowing the **eigenvalues** and **eigenvectors** of a matrix we can easily find its **determinant**, decide whether the matrix has an **inverse** and determine the **powers** of the matrix.
- For an example of linear algebra at work: **Google's search engine**, which relies upon eigenvalues and eigenvectors to **rank** pages with respect to **relevance**.

- Example 1:**

Let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ and $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$Au = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3u$$





- Matrix A **transforms** the vector u by **scalar multiplying** it \Rightarrow the transformation only **changes** the **length** of the vector u unless $\lambda = \pm 1$ (length **unchanged**).
- Note:** the relation $Au = \lambda u$ says that the matrix A applied to u gives a vector in the **same** or **opposite** (negative λ) **direction** of u .
- Definition:** For a **non-zero** vector u in R^n the scalar λ is called an **eigenvalue** of the $n \times n$ matrix A and the vector u is called an **eigenvector** corresponding to λ , which satisfies $Au = \lambda u$.

$$\begin{array}{ccc}
 \text{Eigenvalue} & \downarrow & \\
 & & Au = \lambda u \\
 \text{Eigenvector} & \uparrow & \uparrow
 \end{array}$$

- Example 2: (Verifying eigenvalues and eigenvectors)**

$$\text{Let } A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$\begin{aligned}
 A\mathbf{u} &= \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \overset{\text{Eigenvalue}}{\downarrow} 2 \overset{\text{Eigenvector}}{\uparrow} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\mathbf{u} \\
 A\mathbf{v} &= \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \overset{\text{Eigenvalue}}{\downarrow} 3 \overset{\text{Eigenvector}}{\uparrow} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\mathbf{v}
 \end{aligned}$$

■ **Example 3: (Verifying eigenvalues and eigenvectors)**

$$\text{Let } A = \begin{bmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$A\mathbf{u} = \begin{bmatrix} 5 & 0 & 0 \\ -9 & 4 & -1 \\ -6 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \overset{\text{Eigenvalue}}{\downarrow} 2 \overset{\text{Eigenvector}}{\uparrow} \mathbf{u}$$



- **Theorem 1: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n \times n}$)**
 - (1) An eigenvalue of A is a scalar λ such that $\det(\lambda I - A) = 0$.
 - (2) The eigenvectors of A corresponding to λ are the **nonzero solutions** of the **homogeneous system** $(\lambda I - A)\mathbf{u} = \mathbf{0}$.
- **Note:** $A\mathbf{u} = \lambda\mathbf{u} \Rightarrow (\lambda I - A)\mathbf{u} = \mathbf{0}$ has nonzero solutions iff $\det(\lambda I - A) = 0$.
- The equation $\det(\lambda I - A) = 0$ is called the **characteristic equation** of A .
- The polynomial $p(\lambda) = \det(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0$ is called the **characteristic polynomial** of A .
- **Example 4: (Finding eigenvalues and eigenvectors)**

$$\text{Let } A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

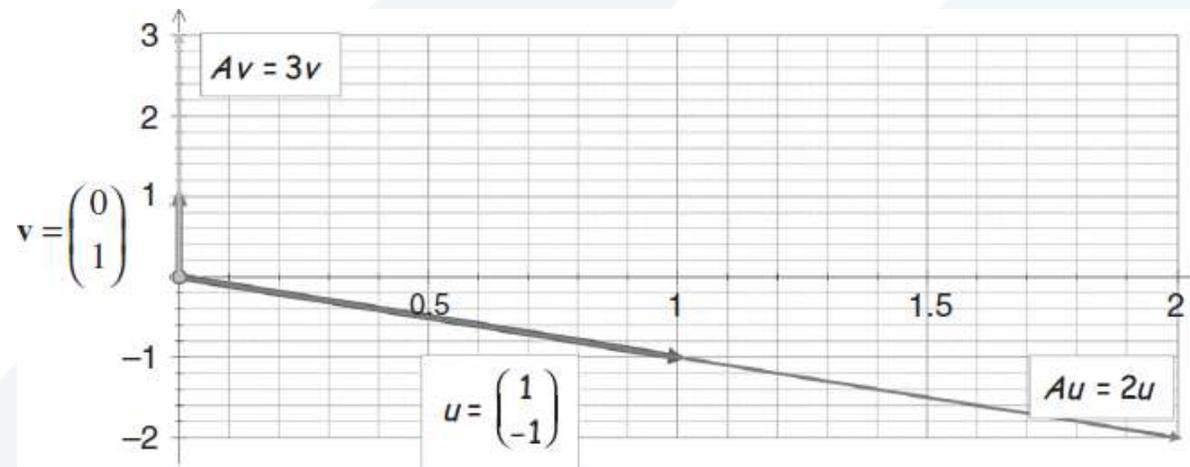


Characteristic equation: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 3) = 0$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = 3$

(1) $\lambda_1 = 2$: $(\lambda_1 I - A)\mathbf{u} = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, t \neq 0$

(2) $\lambda_1 = 3$: $(\lambda_1 I - A)\mathbf{u} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = 0 \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}, t \neq 0$





- Example 5: (Finding eigenvalues and eigenvectors)

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 & -3 \\ -1 & \lambda - 5 & -1 \\ -3 & -1 & \lambda - 1 \end{vmatrix} = (\lambda - 3)(\lambda - 6)(\lambda + 2) = 0$$

Eigenvalues: $\lambda_1 = 3$, $\lambda_2 = 6$ and $\lambda_3 = -2$

$$\lambda_1 = 3: (\lambda I - A)\mathbf{u} = \begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, t \neq 0$$



$$\lambda_2 = 6: (\lambda I - A)\mathbf{u} = \begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = -2: (\lambda I - A)\mathbf{u} = \begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, t \neq 0$$



- Definition:** Let A be an $n \times n$ matrix and have the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. If λ occurs only once then we say λ is a **simple eigenvalue**. If λ occurs m times where $m > 1$ then we say λ is an eigenvalue with **multiplicity** of m .
- In the equation $(\lambda - 1)^3(\lambda - 2) = 0$ we have $\lambda_{1,2,3} = 1$ is an eigenvalue of multiplicity 3 and $\lambda_4 = 2$ is a simple eigenvalue.
- Example 6: (Finding eigenvalues and eigenvectors)**

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & -3 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0 \quad \text{Eigenvalue: } \lambda_{1,2,3} = 2$$



$$\lambda_{1,2,3} = 2: (\lambda I - A)\mathbf{u} = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ -3t \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

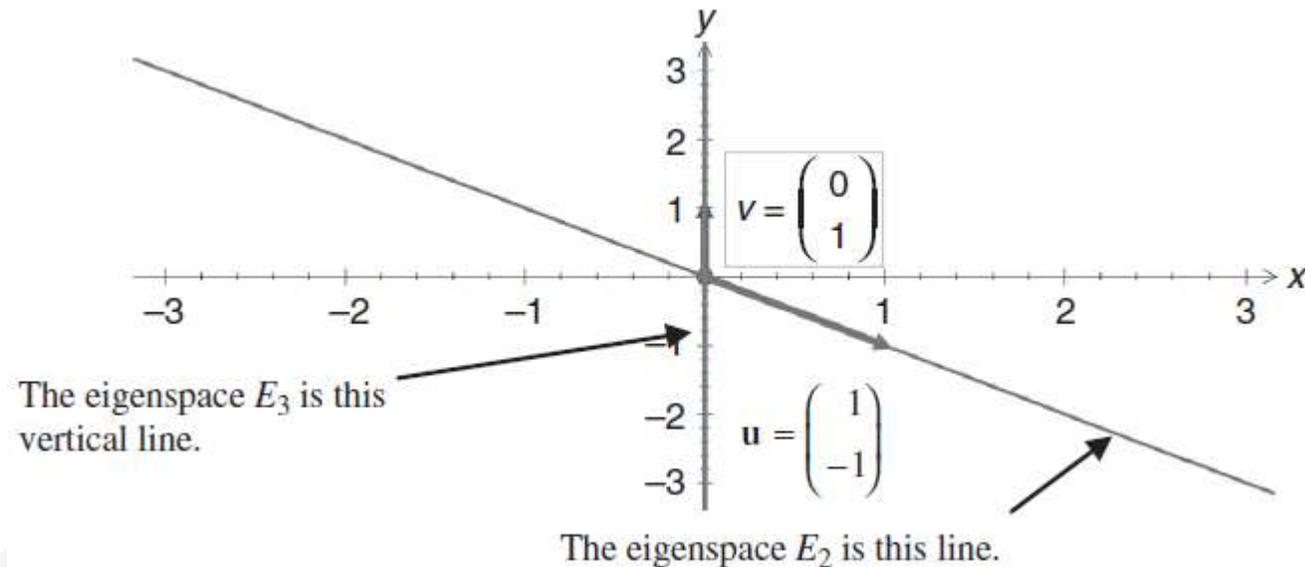
Eigenspace

- If λ is an **eigenvalue** of a square matrix A with an **eigenvector** u then every non-zero **scalar multiplication** of u , is also an eigenvector corresponding to λ .
- If A is an $n \times n$ matrix with an eigenvalue of λ , then the set S of all eigenvectors of A corresponding to λ **together** with the zero vector, 0 , is a subspace of R^n :

$$S = \{0\} \cup \{u | u \text{ is an eigenvector corresponding to } \lambda\}$$



- This subspace S is called an **eigenspace** of λ and is denoted by E_λ , that is $E_\lambda = S$.
- For example, the **eigenspace** associated with **Example 4** for the eigenvalue $\lambda_1 = 2$ is the eigenvector $u = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and for $\lambda_2 = 3$ the eigenvector $v = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

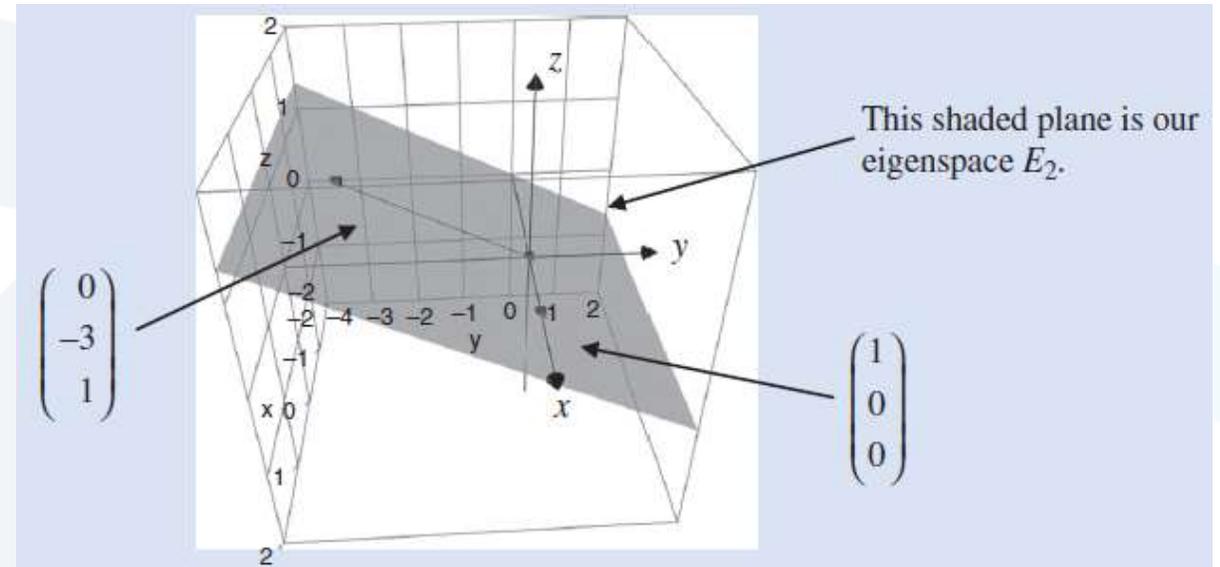


- The **eigenspace** associated with **Example 6** for the eigenvalue $\lambda_{1,2,3} = 2$ is:

$$E_2 = \left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

- Note:** A set of basis vectors B of the eigenspace E_2 are given by:

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$



- Theorem 2: (Eigenvalues of triangular matrices)**

If A is an $n \times n$ diagonal or triangular matrix, then its eigenvalues are the entries on its main diagonal.



- **Example 7: (Finding eigenvalues)**

$$\text{Let } A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}, \quad |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3) = 0$$

$\lambda_1 = 2, \lambda_2 = 1 \text{ and } \lambda_3 = -3$

- **Theorem 3: (Eigenvalues of Powers of a Matrix)**

If A is an $n \times n$ matrix with eigenvector u corresponding to eigenvalue λ .

(1) If m is a positive integer, then λ^m is an eigenvalue of the matrix A^m with the same eigenvector u .

(2) If the matrix A is invertible, then the eigenvalue of the inverse matrix A^{-1} is $1/\lambda$ with the same eigenvector u .

- **Example 8: (Finding eigenvalues)**

Find the eigenvalues of A^3 and A^{-1} of **Example 5**



The eigenvalues of A are: $\lambda_1 = 3$, $\lambda_2 = 6$ and $\lambda_3 = -2$

The eigenvalues of A^3 are: $\lambda_1 = 3^3 = 27$, $\lambda_2 = 6^3 = 216$ and $\lambda_3 = (-2)^3 = -8$

The eigenvalues of A^{-1} are: $\lambda_1 = 1/3$, $\lambda_2 = 1/6$ and $\lambda_3 = -1/2$

- **Theorem 4: (Eigenvalues of triangular matrices)**

If A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

(1) The determinant of the matrix A is given by $\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$.

(2) The trace of the matrix A is given by $\text{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$.

- **Example 9:**

Find the determinant and trace of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ given that the eigenvalues of A are 1, 4 and -1 .

$$\det(A) = (1)(4)(-1) = -4,$$

$$\text{tr}(A) = 1 + 4 - 1 = 4$$



- **Theorem 5: (Eigenvalues and Invertibility)**

A square matrix A is invertible iff $\lambda = 0$ is not an eigenvalue of A

- **Theorem 6: (linearly independent eigenvectors)**

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$. and corresponding eigenvectors p_1, p_2, \dots, p_m where $1 \leq m \leq n$. Then these eigenvectors u_1, u_2, \dots and u_m are linearly independent.

- **Theorem 7: (The Cayley Hamilton Theorem)**

Let A be a square matrix of size $n \times n$, and let $p(\lambda)$ be its characteristic polynomial, i.e. $p(\lambda) = \det(\lambda I - A)$, then:

$$p(A) = A^n + c_{n-1}A^{n-1} + \dots + c_1A + c_0I = O$$



- **Note:** A consequence of the Cayley-Hamilton theorem is a new method for finding the inverse of a nonsingular matrix.

$$A^n + c_{n-1}A^{n-1} + \cdots + c_1A + c_0I = O \Rightarrow A \left[\frac{-1}{c_0} (A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1) \right] = I$$

$$\Rightarrow A^{-1} = \frac{-1}{c_0} (A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1)$$

- **Example 10: (Finding the inverse of a matrix using Cayley-Hamilton theorem)**

Find the inverse of $A = \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}$

$$p(\lambda) = \lambda^3 - 5\lambda^2 + 8\lambda - 4$$

$$\Rightarrow A^{-1} = \frac{1}{4} (A^2 - 5A + 8I)$$

$$A^{-1} = \frac{1}{4} \left(\begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}^2 - 5 \begin{bmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \frac{1}{4} \begin{bmatrix} 18 & 6 & 2 \\ 6 & 20 & 2 \\ 6 & -6 & 4 \end{bmatrix}$$



2. Diagonalization

- **Definition:** If A and B are square matrices, then we say that B is **similar** to A if there is an invertible matrix P such that $B = P^{-1}AP$.
- **Definition:** A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to diagonalize A .
- **Theorem 8: (Similar matrices have the same eigenvalues)**
An $n \times n$ matrix A is **diagonalizable** if and only if it has n **linearly independent** eigenvectors.
- **Steps for diagonalizing an $n \times n$ square matrix:**
Step 1: Find n linearly independent eigenvectors p_1, p_2, \dots, p_n for A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$



Step 2: Let $P = [\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n]$

Step 3: Let $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, where $A\mathbf{p}_i = \lambda_i\mathbf{p}_i$, $i = 1, 2, \dots, n$

▪ **Example 11: (A diagonalizable matrix)**

Show that the matrix A is diagonalizable $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Then find a matrix P such that $P^{-1}AP$ is diagonal.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

Eigenvalues: $\lambda_1 = 4, \lambda_{2,3} = -2$



$$\lambda_1 = 4, \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \lambda_{2,3} = -2, \mathbf{p}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{p}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = [\mathbf{p}_2 \ \mathbf{p}_1 \ \mathbf{p}_3] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P = [\mathbf{p}_2 \ \mathbf{p}_3 \ \mathbf{p}_1] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$



- **Note:** The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D .
- **Example 12: (A matrix that is not diagonalizable)**

Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0 \quad \text{Eigenvalue: } \lambda_{1,2} = 1$$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A does not have two ($n = 2$) linearly independent eigenvectors, so A is not diagonalizable.



- **Theorem 9: (Sufficient conditions for diagonalization)**

If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- **Example 13: (Determining whether a matrix is diagonalizable)**

Determine whether the matrix $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$ is diagonalizable

A is a triangular matrix, its eigenvalues are the main diagonal entries $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = -3$. These three values are distinct, so A is diagonalizable.

- **Theorem 10: (Powers of matrices)**

If an $n \times n$ matrix A is diagonalizable with $P^{-1}AP = D$ where D is a diagonal matrix then $A^k = PD^kP^{-1}$ where k is a positive integer, and



$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

■ **Example 14: (Powers of matrices)**

Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$. Find A^6 given that $P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

$$A^6 = PD^6P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^6 & 0 & 0 \\ 0 & 2^6 & 0 \\ 0 & 0 & 1^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -62 & 0 & -126 \\ 63 & 64 & 63 \\ 63 & 0 & 127 \end{bmatrix}$$



3. Symmetric Matrices and Orthogonal Diagonalization

- A square matrix A is **symmetric** if it is **equal** to its **transpose**: $A = A^T$.
- **Example 15: (Symmetric matrices and nonsymmetric matrices)**

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$

(symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$

(symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

(nonsymmetric)

- **Theorem 11: (Eigenvalues of symmetric matrices)**

If A is an $n \times n$ symmetric matrix, then the following properties are true:

- (1) A is **diagonalizable**.
- (2) All **eigenvalues** of A are **real**.



(3) If λ is an **eigenvalue** of A with **multiplicity** k , then λ has k **linearly independent eigenvectors**.

- A square matrix P is called **orthogonal** if it is **invertible** and $P^{-1} = P^T$.

- **Example 16: (Orthogonal matrices)**

(a) $P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(b) $P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$ is orthogonal because $P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$

- **Theorem 12: (Properties of orthogonal matrices)**

An $n \times n$ matrix P is orthogonal iff its column vectors form an orthonormal set.



- Example 17: (Orthogonal matrices)

Show that the matrix $P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$ is orthogonal

If P is a orthogonal matrix, then $P^{-1} = P^T \Rightarrow PP^T = I$

$$PP^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\{p_1, p_2, p_3\}$

$$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0, \quad \|p_1\| = \|p_2\| = \|p_3\| = 1$$



- **Theorem 13: (Properties of symmetric matrices)**

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A , then their corresponding eigenvectors u and v are orthogonal.

- **Example 18: (Eigenvectors of a symmetric matrix)**

Show that any two eigenvectors of matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ corresponding to distinct eigenvalues are orthogonal.

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4) = 0$$

\Rightarrow Eigenvalues: $\lambda_1 = 2, \lambda_2 = 4$

$$(1) \lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{u} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad s \neq 0$$



$$(2) \lambda_2 = 4 \Rightarrow \lambda_2 I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{u} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -s \\ s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0 \Rightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal}$$

Orthogonal Diagonalization

- matrix A is **orthogonally diagonalizable** when there exists an **orthogonal** matrix P such that $P^{-1}AP = D$ is **diagonal**.
- Theorem 14: (Fundamental theorem of symmetric matrices)**

Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalues if and only if A is symmetric.



- **Orthogonal diagonalization of a symmetric matrix:**

Let A be an $n \times n$ symmetric matrix.

- (1) Find all **eigenvalues** of A and determine the **multiplicity** of each.
- (2) For each eigenvalue of multiplicity 1, choose a **unit eigenvector**. (Find any eigenvector and then normalize it).
- (3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k **linearly independent** eigenvectors. If this set is not orthonormal, apply **Gram-Schmidt** orthonormalization process.
- (4) Steps 2 and 3 produces an **orthonormal** set of n eigenvectors. Use these eigenvectors to form the columns of P . The matrix $P^{-1}AP = P^TAP = D$ will be **diagonal**.



■ **Example 19: (Orthogonal diagonalization)**

Find a matrix P that orthogonally diagonalizes $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$

$$(1) |\lambda I - A| = (\lambda - 3)^2(\lambda + 6) = 0$$

Eigenvalues: $\lambda_1 = -6$, $\lambda_2 = 3$ (has a multiplicity of 2)

$$(2) \lambda_1 = -6, \quad v_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$(3) \lambda_2 = 3, \quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

linearly Independent



Gram-Schmidt Process:

$$w_2 = v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = \begin{bmatrix} -\frac{2}{5} \\ \frac{4}{3} \\ 1 \end{bmatrix}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \quad u_3 = \frac{w_3}{\|w_3\|} = \begin{bmatrix} -\frac{2}{3\sqrt{5}} \\ \frac{4}{3\sqrt{5}} \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$(4) P = [p_1 \ p_2 \ p_3] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} \Rightarrow P^{-1}AP = P^T AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$



4. Singular Value Decomposition

- The **singular value decomposition** (SVD) is one of the most important factorizations of a matrix. SVD factorization can be applied to **any matrix**; it does not need to be a **square** or **symmetric** matrix.
- **Theorem 15:**
If A is an $m \times n$ matrix, then:
 - (a) $A^T A$ is orthogonally diagonalizable.
 - (b) The eigenvalues of $A^T A$ are nonnegative.
- **Definition:** If A is an $m \times n$ matrix, and if $\lambda_1, \lambda_2, \dots, \lambda_n$ are the **eigenvalues** of $A^T A$, then the numbers $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \dots, \sigma_n = \sqrt{\lambda_n}$ are **called the singular values** of A .



- **Example 20: (Singular Values)**

Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial of $A^T A$ is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$

So the eigenvalues of $A^T A$ are: $\lambda_1 = 3$, $\lambda_2 = 1$, and the singular values of A are: $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = 1$



■ **Theorem 16: (Singular Value Decomposition)**

If A is an $m \times n$ matrix of **rank** k , then A can be expressed in the form $A = U \Sigma V^T$, where Σ has size $m \times n$ and can be expressed in partitioned form as:

$$\Sigma = \begin{bmatrix} D & \vdots & 0_{k \times (n-k)} \\ \hline 0_{(m-k) \times k} & \vdots & 0_{(m-k) \times (n-k)} \end{bmatrix}$$

in which D is a **diagonal** $k \times k$ matrix whose successive entries are the first k singular values of A in **nonincreasing** order ($\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_k > 0$), U is an $m \times m$ **orthogonal** matrix, and V is an $n \times n$ **orthogonal** matrix.

$$\mathbf{u}_i = \frac{1}{\sigma_i} A \mathbf{v}_i \quad (i = 1, 2, \dots, k)$$

$V = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]$ orthogonally diagonalizes $A^T A$

$\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}, \dots, \mathbf{u}_m\}$ is an extension of $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$

to an orthonormal basis for R^m .

$$D = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_k \end{bmatrix}$$



- Example 21: (Singular Value Decomposition if A Is Not Square)**

Find a singular value decomposition of the matrix of **Example 20**

The eigenvalues of $A^T A$ are: $\lambda_1 = 3$, $\lambda_2 = 1$, and the singular values of A are:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}, \quad \sigma_2 = \sqrt{\lambda_2} = 1$$

The unit eigenvectors corresponding to λ_1 and λ_2 are

$$\mathbf{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \Rightarrow V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \quad V \text{ orthogonally diagonalizes } A^T A$$

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A \mathbf{v}_1 = \frac{\sqrt{3}}{3} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sigma_2} A \mathbf{v}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

\mathbf{u}_1 and \mathbf{u}_2 are two of the three column vectors of U



To extend the orthonormal set $\{u_1, u_2\}$ to an orthonormal basis for R^3

The vector $u_3 = [x \ y \ z]^T$ needs to be orthogonal to both vectors u_1 and u_2

$$\frac{\sqrt{6}}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \text{and} \quad \frac{\sqrt{2}}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

In matrix form,
$$\begin{bmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$A = U \Sigma V^T$



Reduced Singular Value Decomposition

- The **zero rows** and **columns** of the matrix Σ are superfluous and can be eliminated

$$A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_k \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_k^T \end{bmatrix}$$

which is called a **reduced singular value decomposition RSVD** of A .

$$A = U_1 \Sigma_1 V_1^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_k \mathbf{u}_k \mathbf{v}_k^T$$

which is called a **reduced singular value expansion** of A .

- Example 22: (Reduced Singular Value Decomposition)**
Find a RSVD and a reduced singular value expansion of **Example 21**



Since A has rank 2, with $k = 2$ the reduced SVD of A corresponding to is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T = \sqrt{3} \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} + (1) \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \sqrt{3} \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{6} \end{bmatrix} + (1) \begin{bmatrix} 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \text{ the reduced singular value expansion}$$