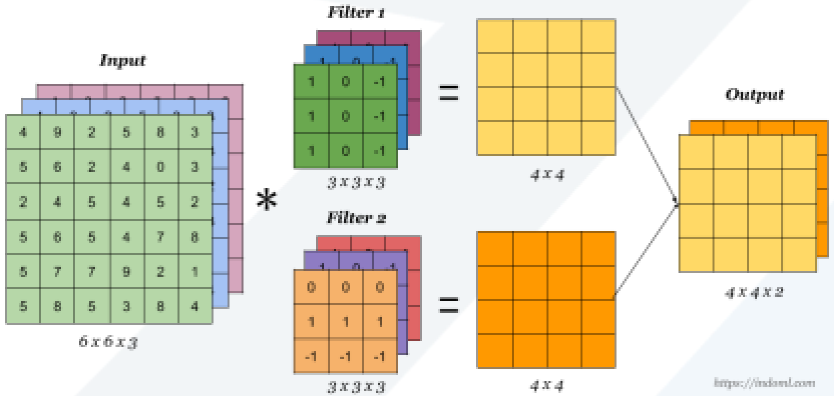
CECC102, CECC122 & CEDC102 : Linear Algebra (and Matrix Theory)

Exercises 8: Eigenvalues and Eigenvectors



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Verify that λ_i is an eigenvalue of A and that x_i is a corresponding eigenvector

① $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad \begin{matrix} \lambda_1 = 2, \boldsymbol{x}_1 = (1,0) \\ \lambda_2 = -2, \boldsymbol{x}_2 = (0,1) \end{matrix}$

$$A\boldsymbol{x}_1 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \lambda_1 \boldsymbol{x}_1$$

$$A\boldsymbol{x}_2 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2 \boldsymbol{x}_2$$

② $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}, \quad \begin{matrix} \lambda_1 = -1, \boldsymbol{x}_1 = (1,1) \\ \lambda_2 = 2, \boldsymbol{x}_2 = (5,2) \end{matrix}$

$$A\boldsymbol{x}_1 = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \boldsymbol{x}_1$$

$$A\boldsymbol{x}_2 = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \lambda_2 \boldsymbol{x}_2$$

③

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}, \quad \begin{matrix} \lambda_1 = 5, \boldsymbol{x}_1 = (1, 2, -1) \\ \lambda_2 = -3, \boldsymbol{x}_2 = (-2, 1, 0) \\ \lambda_3 = -3, \boldsymbol{x}_3 = (3, 0, 1) \end{matrix}$$

$$A\boldsymbol{x}_1 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ -5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \lambda_1 \boldsymbol{x}_1$$

$$A\boldsymbol{x}_2 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 0 \end{bmatrix} = -3\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \lambda_2 \boldsymbol{x}_2$$

$$A\boldsymbol{x}_3 = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ -3 \end{bmatrix} = -3\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \lambda_3 \boldsymbol{x}_3$$

Determine whether x is an eigenvector of A

① $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$

$(a)\ \boldsymbol{x} = (1,2)$

$(b)\ \boldsymbol{x} = (2,1)$

$(c)\ \boldsymbol{x} = (1,-2)$

$(d)\ \boldsymbol{x} = (-1,0)$

$$(a)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ $\boldsymbol{x}$ is not an eigenvector of A

$$(b)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \end{bmatrix} = 8\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue 8)

$$(c)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue 3)

$$(d)\ A\boldsymbol{x} = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 11 \\ 10 \end{bmatrix} \neq \lambda \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$ $\boldsymbol{x}$ is not an eigenvector of A

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② $A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$

$(a)\ \boldsymbol{x} = (2, -4, 6)$
$(b)\ \boldsymbol{x} = (2, 0, 6)$
$(c)\ \boldsymbol{x} = (2, 2, 0)$
$(d)\ \boldsymbol{x} = (-1, 0, 1)$

$$(a)\ A\boldsymbol{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ -16 \\ 24 \end{bmatrix} = 4\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue 4)

$$(b)\ A\boldsymbol{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ -16 \\ 12 \end{bmatrix} \lambda \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$$

$\boldsymbol{x}$ is not an eigenvector of A

$$(c)\ A\boldsymbol{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue -2)

$$(d)\ A\boldsymbol{x} = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\boldsymbol{x}$ is an eigenvector of A (with a corresponding eigenvalue -2)

Find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix

① $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{vmatrix} = \lambda^2 - 7\lambda = \lambda(\lambda - 7) = 0$$

$$(b)\ \lambda_1 = 0, \lambda_2 = 7$$

$$\lambda_1 = 0, \quad \begin{bmatrix} \lambda_1 - 6 & 3 \\ 2 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, 2t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(1, 2)$

$$\lambda_1 = 7, \quad \begin{bmatrix} \lambda_2 - 6 & 3 \\ 2 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(-3t, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 7$ is $(-3, 1)$

② $$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & -1 \\ 0 & \lambda - 3 & -4 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda - 3)(\lambda - 1) = 0$$

$$(b)\ \lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 1$$

$$\lambda_1 = 2,\quad \begin{bmatrix} \lambda_1 - 2 & 0 & -1 \\ 0 & \lambda_1 - 3 & -4 \\ 0 & 0 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, 0, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 2$ is $(1, 0, 0)$

$$\lambda_2 = 3,\quad \begin{bmatrix} \lambda_2 - 2 & 0 & -1 \\ 0 & \lambda_2 - 3 & -4 \\ 0 & 0 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(0, t, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 3$ is $(0,1,0)$

$$\lambda_3 = 1, \quad \begin{bmatrix} \lambda_3 - 2 & 0 & -1 \\ 0 & \lambda_3 - 3 & -4 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(-t, -2t, t) : t \in R\}$. So, an eigenvector corresponding to $\lambda_3 = 1$ is $(-1, -2, 1)$

③ $$A = \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix}$$

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ 2 & \lambda - 5 & 2 \\ 6 & -6 & \lambda + 3 \end{vmatrix} = (\lambda + 3)(\lambda - 3)^2 = 0$$

$(b)\ \lambda_1 = -3,\ \lambda_2 = 3$ (repeated)

$$\lambda_1 = -3,\quad \begin{bmatrix} \lambda_1 - 1 & -2 & 2 \\ 2 & \lambda_1 - 5 & 2 \\ 6 & -6 & \lambda_1 + 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -4 & -2 & 2 \\ 2 & -8 & 2 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, t, 3t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -3$ is (1, 1, 3)

$$\lambda_2 = 3,\quad \begin{bmatrix} \lambda_2 - 1 & -2 & 2 \\ 2 & \lambda_2 - 5 & 2 \\ 6 & -6 & \lambda_2 + 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(s - t, s, t): s, t \in R\}$. So, two eigenvectors corresponding to $\lambda_2 = 3$ are (1,1,0) and (1, 0, –1)

Find (if possible) a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify that $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal

① $A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 6 & 3 \\ 2 & \lambda - 1 \end{vmatrix} = \lambda^2 - 7\lambda = \lambda(\lambda - 7) = 0$$

$$\lambda_1 = 0, \lambda_2 = 7$$

$$\lambda_1 = 0, \quad \begin{bmatrix} \lambda_1 - 6 & 3 \\ 2 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, 2t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(1, 2)$

$$\lambda_2 = 7, \quad \begin{bmatrix} \lambda_2 - 6 & 3 \\ 2 & \lambda_2 - 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$S = \{(-3t, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 7$ is $(-3, 1)$

$$P = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}$$

② $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & 0 & -1 \\ 0 & \lambda-3 & -4 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-2)(\lambda-3)(\lambda-1) = 0$$

$$\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1$$

$$\lambda_1 = 2, \quad \begin{bmatrix} \lambda_1-2 & 0 & -1 \\ 0 & \lambda_1-3 & -4 \\ 0 & 0 & \lambda_1-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(t, 0, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 2$ is $(1, 0, 0)$

$$\lambda_2 = 3, \quad \begin{bmatrix} \lambda_2-2 & 0 & -1 \\ 0 & \lambda_2-3 & -4 \\ 0 & 0 & \lambda_2-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(0, t, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 3$ is $(0, 1, 0)$

$$\lambda_3 = 1, \quad \begin{bmatrix} \lambda_3 - 2 & 0 & -1 \\ 0 & \lambda_3 - 3 & -4 \\ 0 & 0 & \lambda_3 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(-t, -2t, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_3 = 1$ is $(-1, -2, 1)$

$$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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③ $$A=\begin{bmatrix}1 & 2 & -2\\ -2 & 5 & -2\\ -6 & 6 & -3\end{bmatrix}$$

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & -2 & 2\\ 2 & \lambda-5 & 2\\ 6 & -6 & \lambda+3\end{vmatrix}=(\lambda+3)(\lambda-3)^2=0$$

$\lambda_1=-3,\ \lambda_2=3$ (repeated)

$$\lambda_1=-3,\quad \begin{bmatrix}\lambda_1-1 & -2 & 2\\ 2 & \lambda_1-5 & 2\\ 6 & -6 & \lambda_1+3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\Rightarrow\begin{bmatrix}-4 & -2 & 2\\ 2 & -8 & 2\\ 6 & -6 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$S=\{(t,\ t,\ 3t):\ t\in R\}$. So, an eigenvector corresponding to $\lambda_1=-3$ is $(1,\ 1,\ 3)$

$$\lambda_2 = 3,\quad \begin{bmatrix} \lambda_2 - 1 & -2 & 2 \\ 2 & \lambda_2 - 5 & 2 \\ 6 & -6 & \lambda_2 + 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 2 & -2 & 2 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $\{(s - t, s, t): s, t \in R\}$. So, two eigenvectors corresponding to $\lambda_2 = 3$ are (1,1,0) and (1, 0, −1)

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 4 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 4 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ -2 & 5 & -2 \\ -6 & 6 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

④ $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2 = 0$$

$\lambda_1 = 1, \lambda_2 = 2$ (repeated)

$$\lambda_1 = 1, \quad \begin{bmatrix} \lambda_1 - 1 & 0 & 0 \\ -1 & \lambda_1 - 2 & -1 \\ -1 & 0 & \lambda_1 - 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(-t, 0, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 1$ is $(-1, 0, 1)$

$$\lambda_2 = 2, \quad \begin{bmatrix} \lambda_2 - 1 & 0 & 0 \\ -1 & \lambda_2 - 2 & -1 \\ -1 & 0 & \lambda_2 - 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$S = \{(0, t, 0): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 2$ is (0, 1, 0).

There are just two linearly independent eigenvectors of A. So, A is not diagonalizable

Find $\boldsymbol{A}^7$

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

(a) Find eigenvalues of the matrix A

(b) Find eigenvectors for each eigenvalue of A

(c) Diagonalize the matrix A

(d) Calculate A^7

(a) $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -4 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$

$\Rightarrow \lambda_1 = -1, \lambda_2 = 5$

(b) $\lambda_1 = -1, \quad \begin{bmatrix} \lambda_1 - 1 & -2 \\ -4 & \lambda_1 - 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$S = \{(t, -t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = -1$ is $(1, -1)$

$\lambda_2 = 5, \quad \begin{bmatrix} \lambda_2 - 1 & -2 \\ -4 & \lambda_2 - 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & -2 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

The solution is $\{(t, 2t): t \in R\}$. So, an eigenvector corresponding to $\lambda_2 = 5$ is $(1, 2)$

$$(c)\ P=\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix} \Rightarrow P^{-1}=\frac{1}{3}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}$$

$$D=P^{-1}AP=\frac{1}{3}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 5\end{bmatrix}$$

$$(d)\ A=PDP^{-1} \Rightarrow A^7=PD^7P^{-1}$$

$$A^7=\frac{1}{3}\begin{bmatrix}1 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}(-1)^7 & 0\\ 0 & 5^7\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}26041 & 26042\\ 52084 & 52083\end{bmatrix}$$

Determine whether the matrix is orthogonal

① $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

Because the column vectors of the matrix form an orthonormal set, the matrix is orthogonal

② $A = \begin{bmatrix} -4 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 4 \end{bmatrix}$

Because the column vectors of the matrix do not form an orthonormal set [(–4, 0, 3) and (3, 0, 4) are not unit vectors], the matrix is not orthogonal

Show that the matrix below is orthogonal for any value of θ

① $$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A^T$$

Prove that if is an orthogonal matrix, then det (A) = ± 1

If A is orthogonal, then $AA^T = I$. So,

$$1 = \det(AA^T) = \det(A)\det(A^T) = \det(A^2) \Rightarrow \det(A) = \pm 1$$

Find a matrix P such that P^TAP orthogonally diagonalizes A. Verify that P^TAP gives the correct diagonal form

①

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 1 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 2$$

$$\lambda_1 = 0, \begin{bmatrix} \lambda_1 - 1 & -1 \\ -1 & \lambda_1 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$S = \{(t, -t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 0$ is $(1, -1)$

$$\lambda_2 = 2, \begin{bmatrix} \lambda_2 - 1 & -1 \\ -1 & \lambda_2 - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is $\{(t, t): t \in R\}$. So, an eigenvector corresponding to $\lambda_1 = 2$ is (1, 1)

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$P^T A P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

② $$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & -2 \\ 1 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 2 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 4) = 0$$

The eigenvalues are $\lambda_1 = -2$, $\lambda_2 = 2$, and $\lambda_3 = 4$, with corresponding eigenvectors $(-1, -1, 1)$, $(-1, 1, 0)$, and $(1, 1, 2)$, respectively. Normalizing:

$$P = \begin{bmatrix} -\sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{6}/6 \\ -\sqrt{3}/3 & \sqrt{2}/2 & \sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \sqrt{6}/3 \end{bmatrix}$$

$$P^T AP = \begin{bmatrix} -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{3} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{6}}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Solve the system of first-order linear differential equations

1

$$\begin{aligned} y_1' &= \quad\quad\;\; -3y_2 + 5y_3 \\ y_2' &= -4y_1 + 4y_2 - 10y_3 \\ y_3' &= \quad\quad\quad\quad\quad\;\; 4y_3 \end{aligned}$$

$$\boldsymbol{y}' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & -3 & 5 \\ -4 & 4 & -10 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A\boldsymbol{y}$$

The eigenvalues of A are $\lambda_1 = -2$, $\lambda_2 = 6$ and $\lambda_3 = 4$, with corresponding eigenvectors (3, 2, 0), (–1, 2, 0) and (–5, 10, 2), respectively. So, diagonalize A using a matrix P whose column vectors are the eigenvectors of A.

$$P = \begin{bmatrix} 3 & -1 & -5 \\ 2 & 2 & 10 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The solution of the system $\boldsymbol{w}' = P^{-1}AP\boldsymbol{w}$ is $w_1 = C_1 e^{-2t}$, $w_2 = C_2 e^{6t}$, $w_3 = C_3 e^{4t}$.

Return to the original system by applying the substitution $\boldsymbol{y} = P\boldsymbol{w}$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -5 \\ 2 & 2 & 10 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3w_1 - w_2 - 5w_3 \\ 2w_1 + 2w_2 + 10w_3 \\ 2w_3 \end{bmatrix}$$

So, the solution is

$$\begin{aligned} y_1 &= 3C_1 e^{-2t} - C_2 e^{6t} - 5C_3 e^{4t} \\ y_2 &= 2C_1 e^{-2t} + 2C_2 e^{6t} + 10C_3 e^{4t} \\ y_3 &= 2C_3 e^{4t} \end{aligned}$$

Use the Principal Axes Theorem to perform a rotation of axes to eliminate the xy-term in the quadratic equation. Identify the resulting rotated conic and give its equation in the new coordinate system

① $2x^2 + 4xy + 2y^2 + 6\sqrt{2}x + 2\sqrt{2}y + 4 = 0$

The matrix of the quadratic form is $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

This matrix has eigenvalues of 0 and 4, with corresponding unit eigenvectors and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ respectively $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$

Let $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $P^T AP = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$

This implies that the rotated conic is a parabola. Furthermore,

$$[d \quad e]P = [6\sqrt{2} \quad 2\sqrt{2}]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = [4 \quad 8] = [d' \quad e']$$

So, the equation in the $x'y'$-coordinate system is,

$$4(y')^2 + 4x' + 8y' + 4 = 0$$

Determine whether x is an eigenvector of A

① $A = \begin{bmatrix} -3 & 10 \\ 5 & 2 \end{bmatrix}$ (a) $\boldsymbol{x} = (4, 4)$ (b) $\boldsymbol{x} = (-8, 4)$ (c) $\boldsymbol{x} = (-4, 8)$ (d) $\boldsymbol{x} = (5, -3)$

② $A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$ (a) $\boldsymbol{x} = (2, -4, 6)$ (b) $\boldsymbol{x} = (2, 0, 6)$ (c) $\boldsymbol{x} = (2, 2, 0)$ (d) $\boldsymbol{x} = (-1, 0, 1)$

Find (a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix

① $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$ ② $A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ ③ $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

Find (if possible) a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify that $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal

① $A = \begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

③ $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -1 & 1 & 0 \end{bmatrix}$

④ $A = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$

Find A^7

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$

(a) Find eigenvalues of the matrix A

(b) Find eigenvectors for each eigenvalue of A

(c) Diagonalize the matrix A (d) Calculate A^{10}

Determine whether the matrix is orthogonal

① $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

② $A = \begin{bmatrix} -\frac{4}{5} & 0 & \frac{3}{5} \\ 0 & 1 & 0 \\ \frac{3}{5} & 0 & \frac{4}{5} \end{bmatrix}$

Find a matrix P such that P^TAP orthogonally diagonalizes A. Verify that P^TAP gives the correct diagonal form

① $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Solve the system of first-order linear differential equations

① $$\begin{aligned} y_1' &= 6y_1 - y_2 + 2y_3 \\ y_2' &= 3y_2 - y_3 \\ y_3' &= y_3 \end{aligned}$$

Use the Principal Axes Theorem to perform a rotation of axes to eliminate the xy-term in the quadratic equation. Identify the resulting rotated conic and give its equation in the new coordinate system

① $$9x^2 - 24xy + 16y^2 - 400x - 300y = 0$$